LONG TIME ASYMPTOTICS OF HEAT KERNELS AND BROWNIAN WINDING NUMBERS ON MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let M be a compact Riemannian manifold with smooth boundary. We obtain the exact long time asymptotic behaviour of the heat kernel on abelian coverings of M with mixed Dirichlet and Neumann boundary conditions. As an application, we study the long time behaviour of winding numbers of reflected Brownian motions in M. In particular, we prove a Gaussian type central limit theorem showing that when rescaled appropriately, the fluctuation of winding number is normally distributed with an explicit covariance matrix.

1. **Introduction.**

The long time behaviour of Brownian winding number is a classical topic that has been extensively studied by many authors. The first results in this direction appeared in 1958 by Spitzer [Spi58]. In his seminal work, Spitzer proved that the total winding angle of a planar Brownian motion around the origin up to time t, denoted as $\theta(t)$, satisfies the following long time asymptotics:

$$\frac{2\theta(t)}{\log t} \xrightarrow[t\to\infty]{\text{in dist}} \xi,$$

where ξ is the standard Cauchy distribution.

Since the result of Spitzer, many related interesting questions have been studied by various authors. In 1987, two physicists Rudnick and Hu [RH87] (see also Rogers and Williams [RW00]) showed that, if one considers winding of a reflected Brownian motion around a punctured disk instead of a punctured point in the plane, then the limiting distribution becomes of hyperbolic type instead of Cauchy. In addition, it is also natural to consider planar winding around multiple punctured points or disks. However, understanding winding in this context is more complicated due to the fact that the winding number is intrinsically non-abelian (it takes values in a free group) if one wants to keep track of the order of winding around different holes. Abelianized versions of planar Brownian winding around multiple points/disks were studied in [PY86, PY89, GK94, TW95], and various generalizations to the context of positive recurrent diffusions, Riemann surfaces as well as higher dimensions were studied in [GK94, LM84, Wat00]. The techniques used in most of the papers cited above rely on conformal invariance of planar Brownian motion, and hence are specific to two dimensions. There is, however, some literature on more general processes defined by stochastic line integrals. Namely, Kuwada [Kuw09] studied

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large deviation principles and Laplace approximation for current-valued processes in the Riemannian geometric setting.

The present paper is motivated by the following related question: *What is the long time behaviour of the abelianized winding number of normally reflected Brownian motions on compact Riemannian manifolds with boundary?* The simplest example of this is the winding of reflected Brownian motion in a bounded planar domain with holes. Unlike the usual approach based on conformal invariance, we take a more general and geometric viewpoint, which extends to higher dimensions. More precisely, we study the winding of Brownian motion by lifting trajectory onto a suitable abelian covering space, and understand the long time asymptotics by studying the heat kernel. The exact formulation of the problem, and its relation to the study of heat kernels on an associated abelian covering space is stated precisely in Sections 3 and 5.1 respectively. Our result is a refinement of a result by Toby and Werner [TW95], and this is described in Section 5.2.

Motivated by the above question, we are led to a more fundamental question about the long time behaviour of heat kernels on Riemannian manifolds with boundary conditions. The study of heat kernels on Riemannian manifolds lies in one of the core topics in geometric analysis, and it sheds light on many deep connections among analysis, geometry, topology and probability. The short time asymptotics of heat kernels on Riemannian manifolds have been extensively studied and are relatively well understood (see for instance [BGV92, Gri99] and the references therein). The long time asymptotics, on the other hand, are subtly related to global properties of the underlying manifold, and our understanding of this is much less complete.

To our best knowledge, in the Riemannian setting, all scenarios in which the exact long time asymptotics can be determined are summarized as follows. The simplest case is when the underlying Riemannian manifold is compact, with or without boundary. In this case, the long time asymptotics is governed by the bottom spectrum of the Laplace-Beltrami operator. In the non-compact setting, the problem becomes highly non-trivial. Using representation theory, Bougerol [Bou81] determined the exact long time asymptotics for the distribution of K-invariant random walks on a semi-simple Lie group G, and thus for the heat kernel on the symmetric space G/K. Using gradient and Harnack estimates developed by Li and Yau in [LY86], Li [Li86] determined the exact long time asymptotics of heat kernel on Riemannian manifolds with nonnegative Ricci curvature and polynomial volume growth. Using perturbation of twisted Laplacians, Lott [Lot92] and Kotani-Sunada [KS00] determined the exact long time asymptotics of heat kernel on abelian covers of closed Riemannian manifolds. In a more recent paper, using hyperbolic dynamics and mixing properties of geodesic flows in negative curvature, Ledrappier-Lim [LL15] determined the exact long time asymptotics of heat kernel on the universal cover of a negatively curved, closed Riemannian manifold, generalizing the case of spaces with constant negative curvature in which the heat kernel can be written down explicitly.

Motivated by the Brownian winding problem mentioned above, we study abelian covers of compact Riemannian manifolds with smooth boundary, and we impose mixed Dirichlet and Neumann boundary conditions for the Laplacian. Our main result in this part, as stated in Theorem 2.1 below, determines the exact long time asymptotics of the heat kernel on the covering space and the convergence is shown to be uniform. We believe that such result is interesting on its own and is to some extent more fundamental than the original motivation from the winding number problem. Our proof relies on techniques developed in [Lot92, KS00], which is based on representing the heat kernel in terms of a compact family of heat kernels on twisted line bundles and understanding the perturbation of principal eigenvalues for the associated twisted Laplacians. There is, however, one obstruction in using these techniques when Dirichlet boundary conditions are imposed on part of the boundary. Overcoming this difficulty is the main novel contribution of this part of our paper.

Under suitable transformation, the main difficulty mentioned above can be reduced to an elegant eigenvalue minimization problem, that can be stated elementarily as follows. Let ω be a harmonic one form on the base manifold M which is tangential at the boundary. Consider the following eigenvalue problem

$$-\Delta\phi_\omega - 4\pi i \cdot \nabla\phi_\omega + 4\pi^2|\omega|^2\phi_\omega = \mu_\omega\phi_\omega$$

under mixed Dirichlet and Neumann boundary conditions. A key step in our proof requires the following property that appears surprising at first sight: *The eigenvalue μ_ω attains the global minimum if and only if the integral of ω on closed loops is integer-valued. In addition, the minimal is of second order.* This is easily established (and crucially used) in the proofs of [Lot92, KS00] when M has no boundary, or when only Neumann boundary conditions are imposed. In the Dirichlet boundary case, it is the content of Lemma 4.4 and Lemma 4.5 in Section 4.

To summarize, our first main result (Theorem 2.1 in Section 2) shows the heat kernel $\hat{H}(t,x,y)$ on the abelian cover $\hat{M}$ is asymptotically

$$\hat{H}(t,x,y) \approx \frac{C'_{\mathcal{I}}(x,y)}{t^{k/2}} \exp\left(-\lambda_0 t - \frac{d'_{\mathcal{I}}(x,y)^2}{t}\right)$$

as $t \to \infty$ uniformly in $x, y \in \hat{M}$. Here k is the rank of the deck transformation group, λ_0 is the principal eigenvalue of Laplacian on the base manifold M with the specified mixed boundary conditions, and $C'_{\mathcal{I}}, d'_{\mathcal{I}}$ are explicitly defined functions. Our second main result (Theorem 3.2 in Section 3), uses the above heat kernel asymptotics result to prove a Gaussian type central limit theorem for the winding number of normally reflected Brownian motions on compact Riemannian manifolds with smooth boundary.

Plan of this paper. In Section 2, we state our main result concerning the long time asymptotics of the heat kernel on abelian covers of M. In Section 3, we state our main result concerning the long time behaviour of winding of reflected Brownian motion on M. We prove these results in Sections 4 and 5 respectively.

2. Long time behaviour of the heat kernel on abelian covers.

Let M be a compact Riemannian manifold with smooth boundary. Let $\hat{M}$ be a Riemannian cover of M with deck transformation group G and covering map $\boldsymbol{\pi}$. We assume throughout this paper that G is a finitely generated abelian group with rank $k \geqslant 1$, and $M \cong \hat{M}/G$. Let $G_T = \operatorname{tor}(G) \subseteq G$ denote the torsion subgroup of G, and let $G_F \stackrel{\text{def}}{=} G/G_T$. The order of G_T is denoted by $|G_T|$.

Let Δ and $\hat{\Delta}$ be the Laplace-Beltrami operator on M and $\hat{M}$ respectively. Decompose ∂M, the boundary of M, into two pieces $\partial_N M$ and $\partial_D M$, and let $H(t,p,q)$ be the heat kernel of Δ on M with Dirichlet boundary conditions on $\partial_D M$ and Neumann boundary conditions on $\partial_N M$. Let $\partial_D \hat{M} = \boldsymbol{\pi}^{-1}(\partial_D M)$ and

$\boldsymbol{\pi}^{-1}(\partial_N M)$, and let $\hat{H}(t,x,y)$ be the heat kernel of $\hat{\Delta}$ on $\hat{M}$ with Dirichlet boundary conditions on $\partial_D \hat{M}$, and Neumann boundary conditions on $\partial_N \hat{M}$. Let $\lambda_0 \geqslant 0$ be the principal eigenvalue of $-\Delta$ with the given boundary conditions.

Our main result concerning the asymptotic long time behaviour of the heat kernel $\hat{H}$ on the covering space $\hat{M}$ is stated as follows.

Theorem 2.1. *There exist explicit functions $C_{\mathcal{I}}, d_{\mathcal{I}} \colon \hat{M} \times \hat{M} \to [0,\infty)$ (defined in (2.7) and (2.9) below), such that*

$$\lim_{t\to\infty} \Big(t^{k/2} e^{\lambda_0 t} \hat{H}(t,x,y) - \frac{C_{\mathcal{I}}(x,y)}{|G_T|} \exp\Big(-\frac{2\pi^2 d_{\mathcal{I}}^2(x,y)}{t} \Big) \Big) = 0 \,, \tag{2.1}$$

uniformly for $x, y \in \hat{M}$. In particular, for every $x, y \in \hat{M}$, we have

$$\lim_{t\to\infty} t^{k/2} e^{\lambda_0 t} \hat{H}(t,x,y) = \frac{C_{\mathcal{I}}(x,y)}{|G_T|} \,.$$

The definition of the functions $C_{\mathcal{I}}$ and $d_{\mathcal{I}}$ above requires the construction of an inner product structure on a space of harmonic 1-forms over M. More precisely, let

$$\mathcal{H}^1 \stackrel{\text{def}}{=} \{ \omega \in T^*M \mid d\omega = 0,\ d^*\omega = 0,\ \text{and } \omega \cdot \nu = 0 \text{ on } \partial M \} \,,$$

be the space of harmonic 1-forms on M that are tangential on ∂M. Here ν denotes the outward pointing unit normal on ∂M, and depending on the context $x \cdot y$ denotes the dual pairing between co-tangent and tangent vectors, or the inner product given by the Riemannian metric. By the Hodge theorem we know that $\mathcal{H}^1$ is isomorphic to the first de Rham co-homology group on M.

Now define $\mathcal{H}^1_G \subseteq \mathcal{H}^1$ by

$$\mathcal{H}^1_G = \Big\{ \omega \in \mathcal{H}^1 \;\Big|\; \oint_{\hat{\gamma}} \boldsymbol{\pi}^*(\omega) = 0 \text{ for all closed loops } \hat{\gamma} \subseteq \hat{M} \Big\} \,. \tag{2.2}$$

It is easy to see that $\mathcal{H}^1_G$ is naturally isomorphic[1] to $\mathrm{Hom}(G,\mathbb{R})$, and hence $\dim(\mathcal{H}^1_G) = k$. Define an inner product on $\mathcal{H}^1_G$ as follows. Let ϕ_0 be the principal eigenfunction of $-\Delta$ with the given boundary conditions, normalized so that $\phi_0 > 0$ in M and $\|\phi_0\|_{L^2} = 1$. Define the quadratic form $\mathcal{I} \colon \mathcal{H}^1_G \to \mathbb{R}$ by

$$\mathcal{I}(\omega) = 8\pi^2 \int_M |\omega|^2 \phi_0^2 + 8\pi \int_M \phi_0 \omega \cdot \nabla g_\omega \,, \tag{2.3}$$

where g_ω is a[2] solution to the equation

$$-\Delta g_\omega - 4\pi\omega \cdot \nabla \phi_0 = \lambda_0 g_\omega \,, \tag{2.4}$$

[1]The isomorphism between $\mathcal{H}^1_G$ and $\mathrm{Hom}(G;\mathbb{R})$, the dual of the deck transformation group G, can be described as follows. Given $g \in G$, pick a base point $p_0 \in M$, and a pre-image $x_0 \in \boldsymbol{\pi}^{-1}(p_0)$. Now define

$$\varphi_\omega(g) = \int_{x_0}^{g(x_0)} \boldsymbol{\pi}^*(\omega) \,,$$

where the integral is done over any path connecting x_0 and $g(x_0)$. By definition of $\mathcal{H}^1_G$, the above integral is independent of the chosen path. Moreover, since $\boldsymbol{\pi}^*(\omega)$ is the pull-back of ω by the covering projection, it follows that $\varphi_\omega(g)$ is independent of the choice of p_0 or x_0. Thus $\omega \mapsto \varphi_\omega$ gives a canonical homomorphism between $\mathcal{H}^1_G$ and $\mathrm{Hom}(G,\mathbb{R})$. The fact that this is an isomorphism follows from the transitivity of the action of G on fibers.

[2] Note, since λ_0 manifestly belongs to the spectrum of $-\Delta$, the function g_ω is not unique. Moreover, one has to verify a solvability condition to ensure that solutions to equation (2.4) exist. We do this in Lemma 4.5, which is proved in Section 4.4, below.

with boundary conditions

$$g_\omega = 0 \quad \text{on } \partial_D M\,, \qquad \text{and} \qquad \nu \cdot \nabla g_\omega = 0 \quad \text{on } \partial_N M\,. \tag{2.5}$$

In the course of the proof of Theorem 2.1, we will see that $\mathcal{I}$ arises naturally as the quadratic form induced by the Hessian of the principal eigenvalue of a family of elliptic operators (see Lemma 4.5, below).

Using $\mathcal{I}$, we define an inner product on $\mathcal{H}^1_G$ by

$$\langle \omega, \tau \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \frac{1}{4}\big(\mathcal{I}(\omega+\tau) - \mathcal{I}(\omega-\tau)\big), \quad \omega, \tau \in \mathcal{H}^1_G\,.$$

We will show (see Lemma 4.5 below) that the function $\mathcal{I}(\omega)$ is well-defined, and $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ is a positive definite inner product on $\mathcal{H}^1_G$.

Remark 2.2. Under Neumann boundary conditions (i.e. if $\partial_D M = \emptyset$), we know that $\lambda_0 = 0$ and ϕ_0 is constant. In this case, equation (2.4) admits only constant solutions, and $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ is simply the (normalized) L^2-inner product (see also Remark 2.3, below). Under Dirichlet boundary conditions, however, (2.4) admits a non-trivial solutions, and the inner product $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ is not the standard L^2-inner product.

Next, to define the distance function $d_{\mathcal{I}} : \hat{M} \times \hat{M} \to \mathbb{R}$ appearing in Theorem 2.1, we take $x, y \in \hat{M}$ and define $\xi_{x,y} \in (\mathcal{H}^1_G)^* \stackrel{\text{def}}{=} \operatorname{Hom}(\mathcal{H}^1_G; \mathbb{R})$ by

$$\xi_{x,y}(\omega) \stackrel{\text{def}}{=} \int_x^y \boldsymbol{\pi}^*(\omega)\,, \tag{2.6}$$

where the integral is taken over any any smooth path in $\hat{M}$ joining x and y. By definition of $\mathcal{H}^1_G$, the above integral is independent of the choice of path joining x and y. We will show that the function $d_{\mathcal{I}} : \hat{M} \times \hat{M} \to \mathbb{R}$ is given by

$$d_{\mathcal{I}}(x,y) \stackrel{\text{def}}{=} \|\xi_{x,y}\|_{\mathcal{I}^*} = \sup_{\substack{\omega \in \mathcal{H}^1_G, \\ \|\omega\|_{\mathcal{I}} = 1}} \xi_{x,y}(\omega)\,, \quad \text{for } x, y \in \hat{M}\,. \tag{2.7}$$

Here $\|\cdot\|_{\mathcal{I}^*}$ denotes the norm on the dual space $(\mathcal{H}^1_G)^*$ obtained by dualising the inner product $\langle \cdot, \cdot \rangle_{\mathcal{I}}$.

Finally, to define $C_{\mathcal{I}}$, we let

$$\mathcal{H}^1_{\mathbb{Z}} \stackrel{\text{def}}{=} \Big\{ \omega \in \mathcal{H}^1_G \;\Big|\; \oint_\gamma \omega \in \mathbb{Z}, \text{ for all closed loops } \gamma \subseteq M \Big\}\,. \tag{2.8}$$

Clearly $\mathcal{H}^1_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z}^k$, and hence we can find $\omega_1, \dots, \omega_k \in \mathcal{H}^1_{\mathbb{Z}}$ which form a basis of $\mathcal{H}^1_{\mathbb{Z}}$. We will show that $C_{\mathcal{I}}$ is given by

$$C_{\mathcal{I}}(x,y) = (2\pi)^{k/2} \Big| \det\Big(\big(\langle \omega_i, \omega_j \rangle_{\mathcal{I}}\big)_{1 \leqslant i,j \leqslant k} \Big) \Big|^{-1/2} \phi_0(\boldsymbol{\pi}(x)) \phi_0(\boldsymbol{\pi}(y))\,. \tag{2.9}$$

Note that the value of $C_{\mathcal{I}}(x,y)$ doe not depend on the choice of the basis $(\omega_1, \dots, \omega_k)$. Indeed, if $(\omega'_1, \dots, \omega'_k)$ is another such basis of the $\mathbb{Z}$-module $\mathcal{H}^1_{\mathbb{Z}}$, since the change-of-basis matrix belongs to $GL(k, \mathbb{Z})$, it must have determinant ± 1.

We conclude this section by a few remarks on simple and but illustrative cases.

Remark 2.3 (Neumann boundary conditions). If Neumann boundary conditions are imposed on all of ∂M (i.e. $\partial_D M = \emptyset$), then the definitions of $C_{\mathcal{I}}$ and $d_{\mathcal{I}}$ simplify considerably. As mentioned earlier, under Neumann boundary conditions we have

$$\lambda_0 = 0 \qquad \text{and} \qquad \phi_0 \equiv \operatorname{vol}(M)^{-1/2}\,,$$

and hence

$$\langle \omega, \tau \rangle_{\mathcal{I}} = \frac{8\pi^2}{\operatorname{vol}(M)} \int_M \omega \cdot \tau \,, \tag{2.10}$$

is a multiple of the standard L^2-inner product. Here $\omega \cdot \tau$ denotes the inner product on 1-forms inherited from the metric on M. In this case

$$d_{\mathcal{I}}(x, y) = \Big(\frac{\operatorname{vol}(M)}{8\pi^2} \Big)^{1/2} \sup_{\substack{\omega \in \mathcal{H}^1_G \\ \|\omega\|_{L^2(M)} = 1}} \int_x^y \boldsymbol{\pi}^*(\omega) \,,$$

and

$$C_{\mathcal{I}}(x, y) = \frac{(2\pi)^{k/2}}{\operatorname{vol}(M)} \Big| \det \Big((\langle \omega_i, \omega_j \rangle_{\mathcal{I}})_{1 \leqslant i,j \leqslant k} \Big) \Big|^{-1/2}$$

is a constant independent of $x, y \in \hat{M}$.

Note that under Neumann boundary conditions, the heat kernel $\hat{H}(t, x, y)$ on the covering space $\hat{M}$ decays like $t^{-k/2}$ as $t \to \infty$. In contrast, if Dirichlet boundary conditions are imposed on part of the boundary (i.e. $\partial_D M \neq \emptyset$), then $\lambda_0 > 0$ and $\hat{H}(t, x, y)$ decays with rate $t^{-k/2} e^{-\lambda_0 t}$. In addition, in this case $\langle \cdot, \cdot \rangle_{\mathcal{I}}$ is no longer a constant multiple of the standard L^2 inner product.

Remark 2.4 (Computation of ω_i in planar domains). Suppose for now that M is a bounded planar domain with k holes excised and $\operatorname{rank}(G_F) = k$. In this case, the basis $\{\omega_1, \cdots, \omega_k\}$ can be constructed directly by solving some boundary value problems. Indeed, choose (p_j, q_j) inside the j^{th} excised hole and define the harmonic form τ_j by

$$\tau_j \stackrel{\text{def}}{=} \frac{1}{2\pi} \Big(\frac{(p - p_j)\, dq - (q - q_j)\, dp}{(p - p_j)^2 + (q - q_j)^2} \Big) \,. \tag{2.11}$$

Define $\phi_j \colon M \to \mathbb{R}$ to be the solution of the PDE

$$\begin{cases} -\Delta \phi_j = 0 & \text{in } M \,, \\ \partial_\nu \phi_j = \tau_j \cdot \nu & \text{on } \partial M \,. \end{cases}$$

Then ω_j is given by

$$\omega_j = \tau_j + d\phi_j \,.$$

The situation is completely explicit in the case when M is a symmetric annulus (see Example 3.3).

3. The abelianized winding of Brownian motion on manifolds.

We now study the asymptotic behaviour of the abelianized winding of trajectories of reflected Brownian motion on the manifold M using the heat kernel asymptotics given by Theorem 2.1. Although we formulate our result in the geometric setting, the intuition is mostly clear when M is a bounded planar domain with multiple punctured holes. The notion "abelianized" means we are counting the winding number of the Brownian trajectory around each hole but do not keep track of the order of winding around different holes.

The winding of trajectories can be naturally quantified by lifting them to the universal cover. More precisely, let $\bar{M}$ be the universal cover of M, and recall that the fundamental group $\pi_1(M)$ acts on $\bar{M}$ as deck transformations. Fix a fundamental domain $\bar{U} \subseteq \bar{M}$, and for each $g \in \pi_1(M)$ define $\bar{U}_g$ to be the image of $\bar{U}$ under the action of g. Also, define $\bar{\boldsymbol{g}} \colon \bar{M} \to \pi_1(M)$ by

$$\bar{\boldsymbol{g}}(x) = g, \quad \text{if } x \in U_g,$$

to be the map recording which fundamental domain the current position belongs to.

Now given a reflected Brownian motion W in M with normal reflection at the boundary, let $\bar{W}$ be the unique lift of W to $\bar{M}$ starting in $\bar{U}$. Define $\bar{\rho}(t) = \bar{\boldsymbol{g}}(\bar{W}_t) \in \pi_1(M)$. Note that $\bar{\rho}(t)$ measures the (non-abelian) winding of the trajectory of W up to time t.

Our main result of Theorem 2.1 will enable us to study the asymptotic behaviour of the projection of $\bar{\rho}$ to the abelianized fundamental group $\pi_1(M)_{\mathrm{ab}}$. We know that

$$G \stackrel{\text{def}}{=} \pi_1(M)_{\mathrm{ab}} \Big/ \mathrm{tor}(\pi_1(M)_{\mathrm{ab}})$$

is a finitely generated free abelian group, and we let $k = \mathrm{rank}(G)$. Let $\pi_G \colon \pi_1(M) \to G$ be the projection of the fundamental group of M onto G. Fix a choice of loops $\gamma_1, \ldots, \gamma_k \in \pi_1(M)$ so that $\pi_G(\gamma_1), \ldots, \pi_G(\gamma_k)$ form a basis of G.

Definition 3.1. The $\mathbb{Z}^k$-*valued winding number* of W, which is denoted as $\rho(t)$, is the $\mathbb{Z}_k$-valued coordinate process of $\pi_G(\bar{\rho}(t))$ with respect to the basis $\pi_G(\gamma_1), \ldots, \pi_G(\gamma_k)$. Explicitly, $\rho(t) = (\rho_1(t), \ldots, \rho_k(t))$ where

$$\pi_G(\bar{\rho}(t)) = \sum_{i=1}^{k} \rho_i(t) \pi_G(\gamma_i) \,.$$

Note that the $\mathbb{Z}^k$-valued winding number defined above depends on the choice of basis $\gamma_1, \ldots, \gamma_k$. If M is a planar domain with k holes, we can choose γ_i to be a loop that only winds around the i^{th} hole once. In this case, $\rho_i(t)$ is the number of times the trajectory of W winds around the i^{th} hole up to time t.

Our main result concerning the asymptotic long time behaviour of ρ can be stated as follows.

Theorem 3.2. *Let W be a normally reflected Brownian motion in M, and ρ be its $\mathbb{Z}^k$ valued winding number (as in Definition 3.1). Then, there exists a positive definite, explicitly computable covariance matrix Σ (defined in (3.3), below) such that*

$$\frac{\rho(t)}{t} \xrightarrow{p} 0 \qquad \textit{and} \qquad \frac{\rho(t)}{\sqrt{t}} \xrightarrow{w} \mathcal{N}(0, \Sigma) \,. \tag{3.1}$$

Here $\mathcal{N}(0, \Sigma)$ denotes a normally distributed random variable with mean 0 *and covariance matrix Σ.*

We now describe the covariance matrix Σ above. Given $\omega \in \mathcal{H}^1$ define the map $\varphi_\omega \in \mathrm{Hom}(\pi_1(M), \mathbb{R})$ by

$$\varphi_\omega(\gamma) = \int_\gamma \omega \,.$$

It is well known that the map $\omega \mapsto \varphi_\omega$ provides an isomorphism between $\mathcal{H}^1$ and $\mathrm{Hom}(\pi_1(M), \mathbb{R})$. Hence there exists a unique dual basis $\{\omega_1, \ldots, \omega_k\}$ in $\mathcal{H}^1$ such that

$$\int_{\gamma_i} \omega_j = \delta_{i,j} \,. \tag{3.2}$$

The covariance matrix Σ appearing in Theorem 3.2 is given

$$\Sigma_{i,j} \stackrel{\text{def}}{=} \frac{1}{\operatorname{vol} M} \int_M \omega_i \cdot \omega_j \,. \tag{3.3}$$

The proof of Theorem 3.2 follows quite easily from our heat kernel result of Theorem 2.1, which will be given in Section 5 below. We remark, modulo certain amount of technicalities, that Theorem 3.2 can also be proved by using a probabilistic method. We sketch the argument in Section 5.3. To our best knowledge, even in the Euclidean setting, such a result and its proof are not readily available in the literature.

We now mention a few examples where Theorem 3.2 is applicable.

Example 3.3 (An explicit calculation in the annulus). When M is a bounded planar domain with multiple holes, the limiting Gaussian distribution can be computed quite explicitly following Remark 2.4. We consider the simplest case when $M \subseteq \mathbb{R}^2$ is an annulus with inner radius r_1 and outer radius r_2 respectively. In this case, $k = 1$ and $\rho(t)$ is simply the integer-valued winding number of the reflected Brownian motion in M with respect to the inner hole. To define the one form ω_1, choose $p_1 = q_1 = 0$, and define τ_1 by (2.11). Now $\tau_1 \cdot \nu = 0$ on ∂M, forcing $\phi_1 = 0$ and hence $\omega_1 = \tau_1$. Thus Theorem 3.2 shows that $\rho(t)/\sqrt{t} \to \mathcal{N}(0, \sigma^2)$ where

$$\sigma^2 = \frac{1}{\operatorname{vol} M} \int_M |\omega_1|^2 = \frac{1}{2\pi^2 (r_2^2 - r_1^2)} \log\Big(\frac{r_2}{r_1}\Big) \,. \tag{3.4}$$

We remark that in this case Wen [Wen17] proved a finer asymptotic result by explicit calculations:

$$\operatorname{Var}(\rho(t)) \approx \frac{1}{4\pi^2}\Big(\ln^2\big(\frac{r_2}{r_1}\big) - \ln^2\big(\frac{r_1}{r_0}\big)\Big) + \frac{\ln(r_2/r_1)}{2\pi^2(r_2^2 - r_1^2)}\Big(t - \frac{r_2^2 - r_0^2}{2} + r_1^2 \ln\big(\frac{r_2}{r_0}\big)\Big),$$

where $r_0 = |W_0|$ is the radial coordinate of the starting point. Note that Theorem 2.1 only shows $\operatorname{Var}\rho(t)/t \to \sigma^2$ as $t \to \infty$. Wen's result above goes further by providing explicit limit for $\operatorname{Var}\rho(t) - \sigma^2 t$ as $t \to \infty$.

Example 3.4 (Winding in Knot Compliments). Another interesting example is the winding of 3D Brownian motion around knots. Recall that a knot K is an embedding of S^1 into $\mathbb{R}^3$. A basic topological invariant of a knot K is the fundamental group $\pi_1(\mathbb{R}^3 - K)$ which is known as the *knot group* of K. The study of the fundamental group $\pi_1(\mathbb{R}^3 - K)$ is important for the classification of knots and has significant applications in mathematical physics. It is well known that the abelianized fundamental group of $\mathbb{R}^3 - K$ is always cyclic.

Let K be a knot in $\mathbb{R}^3$. Consider the domain $M = \Omega - N_K$, where N is a small tubular neighborhood of K and Ω is a large bounded domain (a ball for instance) containing N_K. Let $W(t)$ be a reflected Brownian motion in M, and define $\rho(t)$ to be the $\mathbb{Z}$-valued winding number of W with respect to a fixed generator of $\pi_1(M)_{\mathrm{ab}}$. Now $\rho(t)$ contains information about the entanglement of $W(t)$ with the knot K. Theorem 3.2 applies in this context, and shows that the long time behaviour of ρ is Gaussian with mean zero and covariance given by (3.3).

In some cases, the generator of $\pi_1(M)_{\mathrm{ab}}$, which is used above to define ρ, can be written down explicitly. For instance, consider the (m,n)-*torus knot*, $K = K_{m,n}$, defined by $S^1 \ni z \mapsto (z^m, z^n) \in S^1 \times S^1$ where $\gcd(m,n) = 1$. Then $\pi_1(M)$ is isomorphic to the free group with two generators a and b, modulo the relation $a^m = b^n$. Here a represents a meridional circle inside the open solid torus and b

represents a longitudinal circle winding around the torus in the exterior. In this case, a generator of $\pi_1(M)_{\text{ab}}$ is $a^{n'}b^{m'}$, where m', n' are integers such that $mm'+nm'=1$. Now $a^{n'}b^{m'}$ represents a unit winding around the knot K, and $\rho(t)$ describes the total number of windings around K.

Finally, we remark that Toby and Werner [TW95] studied the long time asymptotics of the winding number of an obliquely reflected Brownian motion in a bounded planar domain. Under normal reflection with windings around punctured disks, their result becomes a law of large numbers. In this case, our result of Theorem 3.2 is a refinement of Toby and Werner's result, since we are able to show that the long time average of the winding number is zero (see Proposition 5.3 below) and we prove a Gaussian type central limit theorem for fluctuations around the mean. A more detailed discussion about our connection with Toby and Werner's work is presented in Section 5.2 below.

4. Proof of the heat kernel asymptotics (Theorem 2.1).

We follow the main strategy developed by Lott [Lot92] and Kotani-Sunada [KS00], which is based on an integral representation of the heat kernel $\hat{H}(t,x,y)$ on the covering space $\hat{M}$ in terms of a compact family of heat kernels on twisted bundles over the base manifold M. Since M is compact, the long time behaviour of these twisted heat kernels is governed by the principal eigenvalues of the associated twisted Laplacians. It then turns out that the long time behaviour of $\hat{H}(t,x,y)$ can be studied in terms of the behaviour of the above principal eigenvalues near critical point.

In the case when only Neumann boundary conditions are imposed on ∂M (i.e. if $\partial_D M = \emptyset$), the arguments in [Lot92, KS00] can be adapted easily. The main difficulty arises under Dirichlet boundary conditions, largely due to the fact that in this case the principal eigenvalue of Laplacian is strictly positive and the principal eigenfunction is non-constant. One needs to solve a non-trivial eigenvalue minimization problem (Lemma 4.4 and Lemma 4.5 below) concerning the perturbation of principal eigenvalues for the twisted Laplacians, which is almost straight forward in the Neumann boundary case.

Plan of this section. In Section 4.1 we describe the Lott/Kotani-Sunada integral representation of the lifted heat kernel. In Section 4.2 we present the proof of Theorem 2.1 based on the integral representation, assuming the correctness of two lemmas (Lemma 4.4 and Lemma 4.5 below) concerning the eigenvalue minimization problem. Finally in Section 4.3 and Section 4.4, we prove Lemma 4.4 and Lemma 4.5 respectively.

4.1. The Lott/Kotani-Sunanda integral representation of the lifted heat kernel.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle and let $\mathcal{G} \stackrel{\text{def}}{=} \text{Hom}(G, S^1)$ be the space of one dimensional unitary representations of G. We know that $\mathcal{G}$ is a compact Lie group isomorphic to $(S^1)^k$ with a unique normalized Haar measure.

For each given $\chi \in \mathcal{G}$, define an equivalence relation on $\hat{M} \times \mathbb{C}$ by

$$(x, \zeta) \sim (g(x), \chi(g)\zeta) \quad \text{for all } g \in G\,,$$

and let E_χ be the quotient space $\hat{M} \times \mathbb{C}/\sim$. It follows that E_χ is a complex line bundle on M. E_χ carries a natural connection defined by usual differentiation, which together with the Levi-Civita connection on M, induce an associated Laplacian Δ_χ acting on the space $C^\infty(E_\chi)$ of sections of E_χ. If we impose Dirichlet boundary conditions on $\partial_D \hat{M}$ and Neumann boundary conditions on $\partial_N \hat{M}$ respectively, then $-\Delta_\chi$ is a self-adjoint and positive definite elliptic differential operator on $L^2(E_\chi)$.

The above constructions are easily understood from the following viewpoint. First of all, sections of E_χ can be identified with functions $s\colon \hat{M} \to \mathbb{C}$ satisfying the twisting condition

$$s(g(x)) = \chi(g)s(x)\,, \quad \forall x \in \hat{M},\ g \in G\,. \tag{4.1}$$

Define the space

$$\mathcal{D}_\chi \stackrel{\text{def}}{=} \bigl\{ s \in C^\infty(\hat{M}, \mathbb{C}) \bigm| s \text{ satisfies (4.1)}\,,\ s = 0 \text{ on } \partial_D \hat{M}\,, \\ \text{and } \nu \cdot \nabla s = 0 \text{ on } \partial_N \hat{M} \bigr\}\,. \tag{4.2}$$

Then Δ_χ is simply the restriction of the usual Laplacian $\hat{\Delta}$ on $\hat{M}$, and the L^2-inner product is given by

$$\langle s_1, s_2 \rangle_{L^2} \stackrel{\text{def}}{=} \int_M s_1(x_p)\,\overline{s_2(x_p)}\,dp\,, \tag{4.3}$$

for $s_1, s_2 \in \mathcal{D}_\chi$. Here for each $p \in M$, x_p is any point on the fiber $\boldsymbol{\pi}^{-1}(p)$ such that the function $p \mapsto x_p$ is measurable. The twisting condition (4.1) ensures that (4.3) is independent of the choice of x_p.

Remark 4.1. When $\chi \equiv \mathbf{1}$ is the trivial representation, E_χ is exactly the trivial line bundle $M \times \mathbb{C}$, sections of E_χ are just functions on M, and Δ_χ is the standard Laplacian Δ on M.

Let $H_\chi(t, x, y)$ be the heat kernel of $-\Delta_\chi$ on E_χ (see [BGV92] for the general construction of heat kernels on vector bundles). We can view H_χ as a function on $(0, \infty) \times \hat{M} \times \hat{M}$ satisfying the twisting conditions

$$H_\chi(t, g(x), y) = \chi(g)\,H_\chi(t, x, y) \quad \text{and} \quad H_\chi(t, x, g(y)) = \overline{\chi(g)}\,H_\chi(t, x, y)\,.$$

The Lott [Lot92] and Kotani-Sunada [KS00] representation expresses $\hat{H}$ in terms of twisted heat kernels H_χ.

Lemma 4.2 (Lott, Kotani-Sunada)**.** *The heat kernel $\hat{H}$ on $\hat{M}$ satisfies the identity*

$$\hat{H}(t, x, y) = \int_{\mathcal{G}} H_\chi(t, x, y)\,d\chi\,, \tag{4.4}$$

where the integral is performed with respect to the normalized Haar measure $d\chi$ on $\mathcal{G}$.

Proof. Since a full proof can be found in [Lot92, Proposition 38], and [KS00, Lemma 3.1], we only provide a short formal derivation. Suppose $\hat{H}$ is defined by (4.4). Clearly $\hat{H}$ satisfies the heat equation with Dirichlet boundary conditions on $\partial_D \hat{M}$ and Neumann boundary conditions on $\partial_N \hat{M}$). For the initial condition, observe

$$H_\chi(0, x, y) = \sum_{g \in G} \overline{\chi(g)}\,\delta_{g(x)}(y)\,,$$

where $\delta_{g(x)}$ denotes the Dirac delta function at $g(x)$. Integrating over $\mathcal{G}$ and using the orthogonality property

$$\int_{\mathcal{G}} \chi(g)\, d\chi = \begin{cases} 1 & g = \mathrm{Id} \\ 0 & g \neq \mathrm{Id}\,, \end{cases}$$

we see that $\hat{H}(0,x,y) = \delta_x(y)$, and hence $\hat{H}$ must be the heat kernel on $\hat{M}$. $\square$

Remark 4.3. The integral representation (4.4) is similar to Fourier transform and inversion. Indeed, for each $\chi \in \mathcal{G}$, it is easy to see that

$$H_\chi(t,x,y) = \sum_{g \in G} \chi(g) \hat{H}(t,x,g(y))\,.$$

One can view $\mathcal{G} \ni \chi \mapsto H_\chi$ as some sort of Fourier transform of $\hat{H}$, and equation (4.4) gives the inversion formula.

4.2. **Proof of the heat kernel asymptotics (Theorem 2.1).** The representation (4.4) allows us to study the long time behaviour of $\hat{H}$ in terms of the long time behaviour of H_χ, and by the compactness of M, the latter is well known from classical spectral theory. More precisely, the twisted Laplacian Δ_χ admits a sequence of eigenvalues

$$0 \leqslant \lambda_{\chi,1} \leqslant \lambda_{\chi,2} \leqslant \cdots \leqslant \lambda_{\chi,j} \leqslant \cdots \uparrow \infty,$$

and a corresponding sequence of eigenfunctions $\{s_{\chi,j} \mid j \geqslant 0\} \subseteq \mathcal{D}_\chi$ which forms an orthonormal basis of $L^2(E_\chi)$. According to perturbation theory, $\lambda_{\chi,j}$ is smooth in χ, and up to a normalization $s_{\chi,j}$ can be chosen to depend smoothly on χ. The heat kernel $H_\chi(t,x,y)$ can now be written as

$$H_\chi(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_{\chi,j} t} s_{\chi,j}(x) \overline{s_{\chi,j}(y)}\,. \tag{4.5}$$

Note that since M is compact, the above heat kernel expansion is uniform in $x, y \in \hat{M}$ provided the boundary is smooth. This can be seen from the fact that the eigenfunction $s_{\chi,j}$ is uniformly bounded by a polynomial power of eigenvalue $\lambda_{\chi,j}$, together with Weyl's law on the growth the eigenvalues. Combining (4.5) with Lemma 4.2, we have

$$\hat{H}(t,x,y) = \sum_{j=0}^{\infty} \int_{\mathcal{G}} e^{-\lambda_{\chi,j} t} s_{\chi,j}(x) \overline{s_{\chi,j}(y)} d\chi\,. \tag{4.6}$$

From (4.6), it is natural to expect that the long time behaviour of $\hat{H}$ is controlled by the initial term of the series expansion. In this respect, there are two key ingredients for proving Theorem 2.1. The first key point, which is the content of Lemma 4.4, will allow us to see that the integral $\int_{\mathcal{G}} e^{-\lambda_{\chi,0} t} s_{\chi,0}(x) \overline{s_{\chi,0}(y)} d\chi$ concentrates at the trivial representation $\chi = \mathbf{1}$ when t is large. Having such concentration property, the second key point, which is the content of lemma 4.5, will then allow us to determine the long time asymptotics of $\hat{H}$ precisely from the rate at which $\lambda_{\chi,0} \to \lambda_0$ as $\chi \to \mathbf{1} \in \mathcal{G}$. Note that when $\chi = \mathbf{1}$, the corresponding eigenvalue $\lambda_{\mathbf{1},0}$ is exactly λ_0, the principal eigenvalue of $-\Delta$ on M.

Lemma 4.4 (Minimizing the principal eigenvalue)**.** *The function $\chi \mapsto \lambda_{\chi,0}$ attains a unique global minimum on $\mathcal{G}$ at the trivial representation $\chi = \mathbf{1}$.*

We prove Lemma 4.4 in Section 4.3 below. Note that when $\chi = \mathbf{1}$, Δ_χ is simply the standard Laplacian Δ acting on functions on M. If Neumann boundary conditions are imposed on all of ∂M (i.e. when $\partial_D M = \emptyset$), $\lambda_{\mathbf{1},0} = 0$. In this case, the proof of Lemma 4.4 can be adapted from the arguments in [Sun89] (see also a direct proof in Section 4.3 in the Neumann boundary case). However, if Dirichlet boundary conditions are imposed on a portion of ∂M (i.e. $\partial_D M \neq \emptyset$), then $\lambda_{\mathbf{1},0} > 0$ and the proof of Lemma 4.4 requires an entirely different approach.

In view of (4.6) and Lemma 4.4, to determine the long time behaviour of $\hat{H}$, we also need to understand the rate at which $\lambda_{\chi,0}$ approaches the global minimum as $\chi \to \mathbf{1}$. When G is torsion free, the problem can be reduced to the linear space $\mathcal{H}^1_G$. To be precise, $\mathcal{H}^1_G$ can be identified as the Lie algebra of $\mathcal{G}$ in which the exponential map is given by

$$\mathcal{H}^1_G \ni \omega \mapsto \chi_\omega(g) = \exp\Big(2\pi i \int_{x_0}^{g(x_0)} \boldsymbol{\pi}^*(\omega) \in \mathcal{G}\Big)\,, \tag{4.7}$$

where x_0 is some base point and the integral is taken over any smooth path in $\hat{M}$ joining x_0 and $g(x_0)$.

Now the rate at which $\lambda_{\chi,0} \to \lambda_0$ as $\chi \to \mathbf{1} \in \mathcal{G}$ can be obtained from the rate at which $\lambda_{\chi_\omega,0} \to \lambda_0$ as $\omega \to 0 \in \mathcal{H}^1_G$. In fact, as we will see, the quadratic form induced by the Hessian of the map $\omega \mapsto \lambda_{\chi_\omega,0}$ at $\omega = 0$ is precisely $\mathcal{I}(\omega)$ defined by (2.3), and this determines the rate at which $\lambda_{\chi_\omega,0}$ approaches the global minimum λ_0.

Lemma 4.5 (Positivity of the Hessian)**.** *For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |\omega| < \delta$ we have*

$$\Big|\lambda_{\chi_\omega,0} - \lambda_0 - \frac{\mathcal{I}(\omega)}{2}\Big| < \varepsilon \|\omega\|^2_{L^2(M)}\,, \tag{4.8}$$

where $\mathcal{I}(\omega)$ is defined in (2.3). Moreover, the map $\omega \mapsto \mathcal{I}(\omega)$ is a well defined quadratic form, and induces a positive definite inner product on $\mathcal{H}^1_G$.

We point out that the positivity of the quadratic form $\mathcal{I}(\omega)$ is crucial. As mentioned earlier (Remark 2.3), if only Neumann boundary conditions are imposed on ∂M, $\mathcal{I}(\omega)$ is simply a multiple of the standard L^2 inner product on 1-forms over M, and the positivity is straight forward. The main difficulty again lies in the case of Dirichlet boundary conditions, where the positivity is by no mean obvious. We prove Lemma 4.5 in Section 4.4.

Assuming Lemma 4.4 and Lemma 4.5 for the moment, we can now prove Theorem 2.1. In [KS00] for the case without boundary, the authors pointed out that the long time asymptotics is uniformly in $x, y \in \hat{M}$. However, in our modest opinion some essential details seem to be missing. Our main effort in the argument below is devoted to proving uniform convergence. We first consider the case when G is torsion free, and later on show that how the general case can be dealt with from the torsion free case.

Proof of Theorem 2.1 when G is torsion free. Note first that Lemma 4.4 allows us to localize the integral in (4.6) to an arbitrarily small neighborhood of the trivial representation $\mathbf{1}$. More precisely, we claim that for any open neighborhood R of $\mathbf{1} \in \mathcal{G}$, there exist constants $C_1 > 0$, such that

$$\sup_{x,y \in \hat{M}} \Big| e^{\lambda_0 t} \hat{H}(x,y,t) - \int_R \exp\Big(-(\lambda_{\chi,0} - \lambda_0)t\Big) s_{\chi,0}(x) \overline{s_{\chi,0}(y)}\, d\chi \Big| \leqslant e^{-C_1 t}. \tag{4.9}$$

This in particular implies that the long time behaviour of $\hat{H}(t,x,y)$ is determined by the long time behaviour of the integral representation around an arbitrarily small neighborhood of $\mathbf{1} \in \mathcal{G}$.

To establish (4.9), recall that Rayleigh's principle and the strong maximum principle guarantee that $\lambda_{\mathbf{1},0}$ is simple. Standard perturbation theory (c.f. [RS78], Theorem XII.13) guarantees that when χ is sufficiently close to $\mathbf{1}$, the eigenvalue $\lambda_{\chi,0}$ is also simple (i.e. $\lambda_{\chi,0} < \lambda_{\chi,1}$). Now, by Lemma 4.4, we observe

$$\lambda' \stackrel{\text{def}}{=} \min\bigl\{\inf\{\lambda_{\chi,1} \mid \chi \in \mathcal{G}\}\,,\ \inf\{\lambda_{\chi,0} \mid \chi \in \mathcal{G} - R\}\bigr\} > \lambda_0.$$

Hence by choosing $C_1 \in (0, \lambda' - \lambda_0)$, we have

$$\sup_{x,y\in\hat{M}} \Bigl(\Bigl|\sum_{j=1}^{\infty} \int_{\mathcal{G}} e^{-(\lambda_{\chi,j}-\lambda_0)t} s_{\chi,j}(x)\overline{s_{\chi,j}(y)}\, d\chi\Bigr| \\ + \Bigl|\int_{\mathcal{G}-R} e^{-(\lambda_{\chi,0}-\lambda_0)t} s_{\chi,0}(x)\overline{s_{\chi,0}(y)} d\chi\Bigr|\Bigr) \leqslant e^{-C_1 t}$$

for all t sufficiently large. This immediately implies (4.9).

For any small neighborhood R of $\mathbf{1}$ as before, our next task is to convert the integral over R in (4.9) to an integral over a neighborhood of 0 in $\mathcal{H}^1_G$ (the Lie algebra of $\mathcal{G}$) using the exponential map (4.7). To do this, recall $\{\omega_1, \dots, \omega_k\}$ was chosen to be a basis of $\mathcal{H}^1_{\mathbb{Z}} \subseteq \mathcal{H}^1_G$. Identifying $\mathcal{H}^1_G$ with $\mathbb{R}^k$ using this basis, we let $d\omega$ denote the pullback of the Lebesgue measure on $\mathbb{R}^k$ to $\mathcal{H}^1_G$. Equivalently, $d\omega$ is the Haar measure on $\mathcal{H}^1_G$ normalized so that the parallelogram with sides $\omega_1, \dots, \omega_k$ has measure 1. Clearly

$$\int_R \exp\Bigl(-(\lambda_{\chi,0} - \lambda_0)t\Bigr) s_{\chi,0}(x)\overline{s_{\chi,0}(y)}\, d\chi \\ = \int_T \exp\Bigl(-(\mu_\omega - \lambda_0)t\Bigr) s_{\chi_\omega,0}(x)\overline{s_{\chi_\omega,0}(y)}\, d\omega\,, \tag{4.10}$$

where $\mu_\omega \stackrel{\text{def}}{=} \lambda_{\chi_\omega,0}$ and T is the inverse image of R under the map $\omega \mapsto \chi_\omega$.

Recall that the eigenfunctions $s_{\chi_\omega,0}$ appearing above are sections of the twisted bundle E_{χ_ω}. They can be converted to functions on M using some canonical section σ_ω. Explicitly, let $x_0 \in \hat{M}$ be a fixed base point. For given $\omega \in \mathcal{H}^1_G$, define $\sigma_\omega \colon \hat{M} \to \mathbb{C}$ by

$$\sigma_\omega(x) \stackrel{\text{def}}{=} \exp\Bigl(2\pi i \int_{x_0}^{x} \boldsymbol{\pi}^*(\omega)\Bigr)\,, \tag{4.11}$$

where $\boldsymbol{\pi}^*(\omega)$ is the pullback of ω to $\hat{M}$ via the covering projection $\boldsymbol{\pi}$, and the integral is taken along any smooth path in $\hat{M}$ joining x_0 and x. Observe that for any $g \in G$, we have

$$\sigma_\omega(g(x)) = \sigma_\omega(x) \exp\Bigl(2\pi i \int_{x}^{g(x)} \boldsymbol{\pi}^*(\omega)\Bigr) = \chi_\omega(g)\sigma_\omega(x)\,, \tag{4.12}$$

where $\chi_\omega \in \mathcal{G}$ is defined in equation (4.7). Thus σ_ω satisfies the twisting condition (4.1) and hence can be viewed as a section of E_{χ_ω}. Now define

$$\phi_\omega \stackrel{\text{def}}{=} \overline{\sigma_\omega}\, s_{\chi_\omega,0}.$$

Then $\phi_\omega(g(x)) = \phi_\omega(x)$ for all $g \in \mathcal{G}$, and thus ϕ_ω can be viewed as a smooth function on M.

We can now rewrite (4.10) as

$$\int_R \exp\Big(-(\lambda_{\chi,0}-\lambda_0)t\Big) s_{\chi,0}(x)\overline{s_{\chi,0}(y)}\,d\chi \\ = \int_T \exp\Big(-(\mu_\omega-\mu_0)t - 2\pi i \xi_{x,y}(\omega)\Big)\phi_\omega(x)\overline{\phi_\omega(y)}\,d\omega\,. \tag{4.13}$$

where $\xi_{x,y}(\omega)$ is defined in (2.6). Thus, using (4.9), we have

$$\sup_{x,y\in\hat{M}}\Big|e^{\lambda_0 t}\hat{H}(x,y,t) - I_1\Big| \leqslant e^{-C_1 t}\,,\quad \text{for } t \text{ sufficiently large}\,. \tag{4.14}$$

Here

$$I_1 \stackrel{\text{def}}{=} \int_T \exp\big(-(\mu_\omega-\mu_0)t - 2\pi i\xi_{x,y}(\omega)\big)\phi_\omega(x)\overline{\phi_\omega(y)}\,d\omega\,,$$

and C_1 is the constant appearing in (4.9), and depends on the neighborhood R.

By making the neighborhood R (and hence also T) small, we can ensure that ϕ_ω close to ϕ_0. Moreover, when ω is close to 0, Lemma 4.5 implies $\mu_\omega - \mu_0 \approx \mathcal{I}(\omega)/2$. We claim that for any $\eta > 0$, the neighborhood $R \ni \mathbf{1}$ can be chosen such that

$$\limsup_{t\to\infty}\sup_{x,y\in\hat{M}} t^{k/2}(I_1 - I_2) < \eta\,, \tag{4.15}$$

where

$$I_2 \stackrel{\text{def}}{=} \int_{\mathcal{H}^1_G} \exp\Big(-\frac{1}{2}\mathcal{I}(\omega)t - 2\pi i\xi_{x,y}(\omega)\Big)\phi_0(x)\overline{\phi_0(y)}\,d\omega\,.$$

To avoid breaking continuity, we momentarily postpone the proof of (4.15). Now we see that (4.14) and (4.15) combined imply

$$\lim_{t\to\infty}\Big(t^{k/2}e^{\lambda_0 t}\hat{H}(t,x,y) - t^{k/2}I_2\Big) = 0 \tag{4.16}$$

Therefore, to complete the proof, we only need to evaluate I_2 and express it in the form in (2.1).

To do this, write $\omega = \sum c_n\omega_n \in \mathcal{H}^1_G$ and

$$\mathcal{I}(\omega) = \sum_{m,n=1}^k a_{m,n}c_m c_n\,,$$

where $a_{m,n} \stackrel{\text{def}}{=} \langle\omega_m,\omega_n\rangle_{\mathcal{I}}$. Let A be the matrix $(a_{m,n})$, and $a^{-1}_{m,n}$ be the (m,n) entry of the matrix A^{-1}. Then

$$\begin{aligned} I_2 &= \phi_0(x)\,\overline{\phi_0(y)}\cdot \\ &\qquad \int_{c\in\mathbb{R}^k}\exp\Big(-\sum_{m,n=1}^k a_{m,n}c_m c_n t - 2\pi i\sum_{m=1}^k c_m\xi_{x,y}(\omega_m)\Big)\,dc_1\cdots dc_k \\ &= \phi_0(x)\,\overline{\phi_0(y)}\frac{(2\pi)^{k/2}}{t^{k/2}\det(a_{m,n})^{1/2}}\exp\Big(-\frac{2\pi^2}{t}\sum_{m,n=1}^k a^{-1}_{m,n}\xi_{x,y}(\omega_m)\xi_{x,y}(\omega_n)\Big) \\ &= \phi_0(x)\,\overline{\phi_0(y)}\frac{(2\pi)^{k/2}}{t^{k/2}\det(a_{m,n})^{1/2}}\exp\Big(-\frac{2\pi^2}{t}\|\xi_{x,y}\|^2_{\mathcal{I}^*}\Big)\,, \end{aligned}$$

where the second equality follows from the formula for the Fourier transform of Gaussian distribution. Note that ϕ_0 is real, and therefore

$$I_2 = t^{-k/2} C_{\mathcal{I}}(x,y) \exp\Big(-\frac{2\pi^2 d_{\mathcal{I}}^2(x,y)}{t}\Big)\,,$$

where $C_{\mathcal{I}}$ is defined by (2.9). Combined with (4.16), this finishes the proof of Theorem 2.1 when G is torsion free.

It remains to prove (4.15). Since $\omega \mapsto \phi_\omega$ is continuous, there exists a neighborhood $T \ni 0$ such that

$$\sup_{x\in\widehat{M}} \big|\phi_\omega(x) - \phi_0(x)\big| < \eta \quad \text{for all } \omega \in T\,. \tag{4.17}$$

Now we know that (4.14) holds with some constant $C_1 = C_1(\eta) > 0$ when t is large. Write

$$t^{k/2}(I_1 - I_2) = J_1 + J_2 + J_3\,,$$

where

$$J_1 \stackrel{\text{def}}{=} t^{k/2} \int_T \Big(e^{-(\mu_\omega-\mu_0)t} - e^{-\mathcal{I}(\omega)t/2}\Big) \exp\big(-2\pi i \xi_{x,y}(\omega)\big) \phi_\omega(x)\overline{\phi_\omega(y)}\, d\omega\,,$$

$$J_2 \stackrel{\text{def}}{=} t^{k/2} \int_T \exp\Big(-\frac{1}{2}\mathcal{I}(\omega)t - 2\pi i \xi_{x,y}(\omega)\Big)\Big(\phi_\omega(x)\overline{\phi_\omega(y)} - \phi_0(x)\overline{\phi_0(y)}\Big)\, d\omega\,,$$

and

$$J_3 \stackrel{\text{def}}{=} t^{k/2} \int_{\mathcal{H}^1_G - T} \exp\Big(-\frac{1}{2}\mathcal{I}(\omega)t - 2\pi i \xi_{x,y}(\omega)\Big)\phi_0(x)\overline{\phi_0(y)}\, d\omega\,.$$

First, by Lemma 4.5, $\mathcal{I}(\omega)$ is a positive definite quadratic form, and hence the Gaussian tail estimate shows there exists $C_2 = C_2(\eta) > 0$, such that

$$|J_3| \leqslant e^{-C_2 t}$$

uniformly in $x, y \in \hat{M}$, when t is sufficiently large.

Next, by (4.17) and the positivity of the quadratic form $\mathcal{I}(\omega)$, we have

$$|J_2| \leqslant C_3 \eta t^{k/2} \int_T e^{-\mathcal{I}(\omega)t/2}\, d\omega = C_3\eta \int_{\sqrt{t}\cdot T} e^{-\mathcal{I}(v)/2}\, dv \leqslant C_4 \eta\,,$$

uniformly in $x, y \in \hat{M}$.

Finally, to estimate J_1, first choose $K \subseteq \mathcal{H}^1_G$ compact such that

$$\int_{\mathcal{H}^1_G - K} \exp\Big(-\frac{1}{4}\mathcal{I}(v)\Big)\, dv < \eta\,.$$

By using the same change of variables $v = \sqrt{t}\omega$, we write

$$J_1 = J_1' + J_1'',$$

where

$$J_1' \stackrel{\text{def}}{=} \int_K \Big(\exp\Big(-\Big(\mu_{v/t^{1/2}} - \mu_0\Big)t\Big) - \exp\Big(-\frac{1}{2}\mathcal{I}(v)\Big)\Big) \cdot \exp\Big(-\frac{2\pi i}{\sqrt{t}}\xi_{x,y}(v)\Big)\phi_{v/t^{1/2}}(x)\overline{\phi_{v/t^{1/2}}(y)}\, dv$$

and

$$J_1'' \stackrel{\text{def}}{=} \int_{\sqrt{t}\cdot T - K} \Big(\exp\Big(-\Big(\mu_{v/t^{1/2}} - \mu_0\Big)t\Big) - \exp\Big(-\frac{1}{2}\mathcal{I}(v)\Big)\Big)$$

$$\cdot \exp\Big(-\frac{2\pi i}{\sqrt{t}}\xi_{x,y}(v)\Big)\phi_{v/t^{1/2}}(x)\overline{\phi_{v/t^{1/2}}(y)}\,dv$$

respectively. By Lemma 4.5, we know that

$$\lim_{t\to\infty}\big(\mu_{v/t^{1/2}}-\mu_0\big)t=\frac{1}{2}\mathcal{I}(v)\,,$$

for every $v\in\mathcal{H}^1_G$. Therefore, by the dominated convergence theorem, we have

$$\lim_{t\to\infty}\sup_{x,y\in\hat{M}}|J_1'|=0\,.$$

To estimate J_1'', choose $\varepsilon>0$ such that

$$\frac{1}{4}\mathcal{I}(\omega)\geqslant\varepsilon\|\omega\|^2_{L^2(M)}\,,\qquad\text{for all }\omega\in\mathcal{H}^1_G\,.$$

For this ε, Lemma 4.5 allows us to further assume that T is small enough so that

$$\omega\in T\implies\mu_\omega-\mu_0\geqslant\frac{1}{2}\mathcal{I}(\omega)-\varepsilon\|\omega\|^2_{L^2(M)}\geqslant\frac{1}{4}\mathcal{I}(\omega).$$

In particular, we have

$$v\in\sqrt{t}\cdot T\implies\big(\mu_{v/t^{1/2}}-\mu_0\big)t\geqslant\frac{1}{4}\mathcal{I}(v).$$

It follows that

$$\begin{aligned}
J_1''&\leqslant C_5\int_{\sqrt{t}\cdot T-K}\Big(\exp\big(-(\mu_{v/t^{1/2}}-\mu_0)t\big)+\exp\Big(-\frac{1}{2}\mathcal{I}(v)\Big)\Big)\,dv\\
&\leqslant 2C_5\int_{\sqrt{t}\cdot T-K}\exp\Big(-\frac{1}{4}\mathcal{I}(v)\Big)\,dv\\
&\leqslant 2C_5\int_{\mathcal{H}^1_G-K}\exp\Big(-\frac{1}{4}\mathcal{I}(v)\Big)dv\\
&\leqslant 2C_5\eta\,,
\end{aligned}$$

uniformly in $x,y\in\hat{M}$.

Combining the previous estimates, we conclude

$$\overline{\lim_{t\to\infty}}\sup_{x,y\in\hat{M}}\Big(t^{k/2}(I_1-I_2)\Big)\leqslant(C_4+2C_5)\eta\,,$$

and η with $\eta/(C_4+2C_5)$ yields (4.15) as claimed. $\square$

When G is has a torsion subgroup, we prove Theorem 2.1 by factoring through an intermediate finite cover.

Proof of Theorem 2.1 when G has a torsion subgroup. Since G can be (non-canonically) expressed as a direct sum $G_T\oplus G_F$, we define $M_1=\hat{M}/G_F$. This leads to the covering factorization

$$\begin{tikzcd}
\hat{M} \arrow[r, "\boldsymbol{\pi}_F"] \arrow[dr, "\boldsymbol{\pi}"'] & M_1 \overset{\text{def}}{=} \hat{M}/G_F \arrow[d, "\boldsymbol{\pi}_T"] \\
 & M\,,
\end{tikzcd}\tag{4.18}$$

where $\boldsymbol{\pi}_T$ and $\boldsymbol{\pi}_F$ have deck transformation groups G_T and G_F respectively, and M_1 is compact.

Recall that λ_0 is the principal eigenvalue of $-\Delta$ on M, and ϕ_0 is the corresponding L^2 normalized eigenfunction. Let Λ_0 be the principal eigenvalue of $-\Delta_1$ on M_1, and Φ_0) be the corresponding L^2 normalized eigenfunction. (Here Δ_1 is the Laplacian on M_1.)

Notice that $\boldsymbol{\pi}_T^*\phi_0$, the pull back of ϕ_0 to M_1, is an eigenfunction of $-\Delta_1$ and $\|\boldsymbol{\pi}_T^*\phi_0\|_{L^2(M)} = |G_T|^{1/2}$. Thus

$$\Lambda_0 = \lambda_0 \qquad \text{and} \qquad \Phi_0 = \frac{\boldsymbol{\pi}_T^*\phi_0}{|G_T|^{1/2}}\,. \tag{4.19}$$

Let $\mathcal{I}_1(\omega_1)$ be the analogue of $\mathcal{I}$ (defined in equation (2.3)) for the manifold M_1. Explicitly,

$$\mathcal{I}_1(\omega_1) = 8\pi^2\int_{M_1} |\omega_1|^2\Phi_0^2 + 8\pi\int_{M_1} \Phi_0\,\omega_1\cdot\nabla g_1\,,$$

where g_1 is a solution of

$$-\Delta g_1 - 4\pi\omega_1\cdot\nabla\Phi_0 = \Lambda_0 g_1\,,$$

with Dirichlet boundary conditions on $\boldsymbol{\pi}_T^{-1}(\partial_D M)$ and Neumann boundary conditions on $\boldsymbol{\pi}_T^{-1}(\partial_N M)$. Note that given $\omega_1 \in \mathcal{H}^1_G(M_1)$ we can find $\omega \in \mathcal{H}^1_G(M)$ such that $\boldsymbol{\pi}_T^*(\omega) = \omega_1$. Indeed, since $\dim(\mathcal{H}^1_G(M)) = \dim(\mathcal{H}^1_G(M_1)) = k$ and $\boldsymbol{\pi}_T^*\colon \mathcal{H}^1_G(M) \to \mathcal{H}^1_G(M_1)$ is injective linear map, it must be an isomorphism.

Now using (4.19), we observe that up to an addition of a scalar multiple of Φ_0, we have

$$g_1 = \frac{\boldsymbol{\pi}_T^* g}{|G_T|^{1/2}}\,,$$

where $g = g_\omega$ is defined in (2.4). Therefore,

$$\begin{aligned}\mathcal{I}_1(\omega_1) &= 8\pi^2|G_T|\int_M |\omega|^2\frac{\phi_0^2}{|G_T|} + 8\pi|G_T|\int_M \frac{\phi_0}{|G_T|^{1/2}}\omega\cdot\nabla\Big(\frac{g}{|G_T|^{1/2}}\Big)\\ &= 8\pi^2\int_M |\omega|^2\phi_0^2 + 8\pi\int_M \phi_0\,\omega\cdot\nabla g = \mathcal{I}(\omega)\,.\end{aligned} \tag{4.20}$$

Since the deck transformation group of $\hat{M}$ as a cover of M_1 is torsion free, we can apply Theorem 2.1 to M_1. Thus, we have

$$\lim_{t\to\infty}\Big(t^{k/2}e^{\Lambda_0 t}\hat{H}(t,x,y) - C_{\mathcal{I}_1}(x,y)\exp\Big(-\frac{2\pi^2 d^2_{\mathcal{I}_1}(x,y)}{t}\Big)\Big) \tag{4.21}$$

uniformly on $\hat{M}$. Now using (4.19) and (4.20), we see that

$$d_{\mathcal{I}_1} = d_{\mathcal{I}}, \quad C_{\mathcal{I}_1}(x,y) = \frac{1}{|G_T|}C_{\mathcal{I}}(x,y)\,,$$

and hence the proof is complete. □

The rest of this section is devoted to the proofs of the two key Lemmas 4.4 and 4.5 respectively.

4.3. **Minimizing the principal eigenvalue (proof of Lemma 4.4).** Our aim in this subsection is to prove Lemma 4.4, which asserts that the function $\chi \mapsto \lambda_{\chi,0}$ attains a unique global minimum at $\chi = \mathbf{1}$. The Neumann boundary case is conceptually simpler and we first provide an independent proof for this case. The full proof of Lemma 4.4 under mixed Dirichlet and Neumann boundary conditions will be given later.

Proof of Lemma 4.4 under Neumann boundary conditions. In this case we know that $\lambda_0 = \lambda_{\mathbf{1},0} = 0$, and the corresponding eigenfunction $s_{\mathbf{1},0}$ is constant. Therefore, to prove the lemma, it suffices to show that $\lambda_{\chi,0} > 0$ for all $\chi \neq \mathbf{1}$.

To see this, given $\chi \in \mathcal{G}$, let $s = s_{\chi,0} \in \mathcal{D}_\chi$ be the principal eigenfunction of $-\Delta_\chi$, and $\lambda = \lambda_{\chi,0}$ be the principal eigenvalue. We claim that for any fundamental domain $U \subseteq \hat{M}$, the eigenvalue λ satisfies

$$\lambda \int_U |s|^2 \, dx = \int_U |\nabla s|^2 \, dx \,. \tag{4.22}$$

Once (4.22) is established, it is immediate that $\lambda > 0$ when $\chi \neq \mathbf{1}$. Indeed, if $\chi \neq \mathbf{1}$, $s(g(x)) = \chi(g)s(x)$ forces the function s to be non-constant, and now equation (4.22) forces $\lambda > 0$.

To prove (4.22) observe

$$\lambda \int_U |s|^2 = -\int_U \bar{s}\Delta_\chi s = \int_U |\nabla s|^2 - \int_{\partial U} \bar{s}\,\partial_\nu s \,. \tag{4.23}$$

Here, $\partial_\nu s = \nu \cdot \nabla s$ is the outward pointing normal derivative on ∂U. We will show that the twisting condition (4.1) ensures that the boundary integral above vanishes.

Decompose ∂U as

$$\partial U = \Gamma_1 \cup \Gamma_2 \,, \quad \text{where } \Gamma_1 \stackrel{\text{def}}{=} \partial U \cap \partial \hat{M}, \quad \text{and } \Gamma_2 \stackrel{\text{def}}{=} \partial U - \Gamma_1 \,.$$

Note Γ_1 is the portion of ∂U contained in $\partial \hat{M}$, and Γ_2 is the portion of ∂U that is common to neighboring fundamental domains. Clearly, the Neumann boundary condition (4.24) implies

$$\int_{\Gamma_1} \bar{s}\,\partial_\nu s = 0 \,.$$

For the integral over Γ_2, let $(e_1, \dots, e_k)$ be a basis of G and note that Γ_2 can be expressed as the disjoint union

$$\Gamma_2 = \bigcup_{j=1}^k \bigl(\Gamma_{2,j}^+ \cup \Gamma_{2,j}^-\bigr) \,,$$

where the $\Gamma_{2,j}^\pm$ are chosen so that $\Gamma_{2,j}^+ = e_j(\Gamma_{2,j}^-)$. Using the twisting condition (4.1) and the fact that the action of e_j reverses the direction of the unit normal on $\Gamma_{2,j}^-$, we see

$$\begin{aligned}
\int_{\Gamma_{2,j}^+} \overline{s(x)}\,\partial_\nu s(x)\,dx &= -\int_{\Gamma_{2,j}^-} \overline{s\bigl(e_j(y)\bigr)}\,\partial_\nu s\bigl(e_j(y)\bigr)\,dy \\
&= -\int_{\Gamma_{2,j}^-} \overline{\chi(e_j)}\chi(e_j)\,\overline{s(y)}\,\bigl(\partial_\nu s(y)\bigr)\,dy \\
&= -\int_{\Gamma_{2,j}^-} \overline{s(y)}\,\partial_\nu s(y)\,dy \,,
\end{aligned}$$

Consequently,

$$\int_{\Gamma_2} \bar{s}\,\partial_\nu s = \sum_{j=1}^k \Bigl(\int_{\Gamma_{2,j}^+} + \int_{\Gamma_{2,j}^-}\Bigr)\bar{s}\,\partial_\nu s = 0 \,.$$

and hence the boundary integral in (4.23) vanishes. Thus (4.22) holds, and the proof is complete. $\square$

In the general case when $\partial_D M \neq \emptyset$, $\lambda_{\chi,0} > 0$ for every $\chi \in \mathcal{G}$, and all eigenfunctions are non-constant. This causes the previous argument to break down and the proof involves a different idea. Before beginning the proof, we first make use of a canonical section to transfer the problem to the linear space $\mathcal{H}^1_G$.

Let Ω be the space of $\mathbb{C}$-valued smooth functions $f\colon M \to \mathbb{C}$ such that $f = 0$ on $\partial_D M$ and $\langle \nabla f, \nu \rangle = 0$ on $\partial_N M$. Let $\hat{f} = f \circ \boldsymbol{\pi}\colon \hat{M} \to \mathbb{C}$. Now given $\omega \in \mathcal{H}^1_G$, let σ_ω (defined in (4.12)) be the canonical section and $\chi_\omega \in \mathcal{G}$ be the exponential as defined in (4.7). Notice that the function $\sigma_\omega \hat{f} \in \mathcal{D}_{\chi_\omega}$ is a section on E_{χ_ω}. Clearly $\sigma_\omega \hat{f} = 0$ on $\partial_D \hat{M}$. Moreover, since $\omega \cdot \nu = 0$ on ∂M we have

$$\nu \cdot \nabla \sigma_\omega = 0 \qquad \text{on } \partial \hat{M}\,. \tag{4.24}$$

and hence $\nu \cdot \nabla(\sigma_\omega \hat{f}) = 0$ on $\partial_N \hat{M}$. Thus $\sigma_\omega \hat{f} \in \mathcal{D}_{\chi_\omega}$, where $\mathcal{D}_{\chi_\omega}$ is defined in equation (4.2), and the map $f \mapsto \hat{f}\sigma_\omega$ defines a unitary isomorphism between $\Omega \subseteq L^2(M)$ and $\mathcal{D}_{\chi_\omega} \subseteq L^2(E_{\chi_\omega})$ respecting the imposed boundary conditions.

Now, since ω and $\hat{\omega} \stackrel{\text{def}}{=} \omega \circ \boldsymbol{\pi}$ are both harmonic, we compute

$$\Delta_{\chi_\omega}(\hat{f}\sigma_\omega) = ((H_\omega f) \circ \boldsymbol{\pi})\, \sigma_\omega\,,$$

where H_ω is the self-adjoint operator on $\Omega \subseteq L^2(M)$ defined by

$$H_\omega f \stackrel{\text{def}}{=} \Delta f + 4\pi i\, \omega \cdot \nabla f - 4\pi^2 |\omega|^2 f\,. \tag{4.25}$$

Here we used the Riemannian metric to identify the 1-form ω with a vector field.

The above shows that Δ_{χ_ω} is unitarily equivalent to H_ω. In particular, eigenvalues of $-H_\omega$, denoted by $\mu_{\omega,j}$ are exactly $\lambda_{\chi_\omega,j}$, the eigenvalues of $-\Delta_{\chi_\omega}$. Moreover, the corresponding eigenfunctions, denoted by $\phi_{\omega,j}$, are given by

$$\phi_{\omega,j} = \frac{s_{\chi_\omega,j}}{\sigma_\omega}\,, \quad j \geqslant 0\,. \tag{4.26}$$

Note that $\phi_{\omega,j}$ is a well-defined function on M that satisfies Dirichlet boundary conditions on $\partial_D M$ and Neumann boundary conditions on $\partial_N M$.

We will now prove the general case of Lemma 4.4 by minimizing eigenvalues of the operator $-H_\omega$.

Proof of Lemma 4.4. Let $\omega \in \mathcal{H}^1_G$ and let $\chi_\omega = \exp(\omega) \in \mathcal{G}$ be the corresponding representation defined by (4.7). Let $\mu_\omega = \mu_{\omega,0} = \lambda_{\chi_\omega,0}$ and $\phi_\omega = \phi_{\omega,0}$ where $\phi_{\omega,0}$ is the principal eigenfunction of $-H_\omega$ as defined in (4.26) above. Using (4.25) we see

$$-\Delta\phi_\omega - 4\pi i \omega \cdot \nabla \phi_\omega + 4\pi^2 |\omega|^2 \phi_\omega = \mu_\omega \phi_\omega\,, \tag{4.27}$$

$$-\Delta \phi_0 = \mu_0 \phi_0\,, \tag{4.28}$$

with Dirichlet boundary conditions on $\partial_D \hat{M}$ and Neumann boundary conditions on $\partial_N \hat{M}$. Here μ_0 and ϕ_0 denote the principal eigenvalue and eigenfunction respectively when $\omega \equiv 0$. Note that when $\omega \in \mathcal{H}^1_{\mathbb{Z}}$, the corresponding representation χ_ω is the trivial representation $\mathbf{1}$. We will show that μ_ω above achieves a global minimum precisely when $\omega \in \mathcal{H}^1_{\mathbb{Z}}$ and $\chi_\omega = \mathbf{1}$.

Now let $\varepsilon > 0$ and write

$$\overline{\phi_\omega} = (\phi_0 + \varepsilon) f \quad \text{where } f \stackrel{\text{def}}{=} \frac{\overline{\phi_\omega}}{\phi_0 + \varepsilon}\,.$$

Multiplying both sides of (4.27) by $\overline{\phi_\omega} = (\phi_0 + \varepsilon) f$ and integrating over M gives

$$\begin{aligned}
-\int_M (\Delta\phi_\omega)(\phi_0+\varepsilon) f &= \int_M \nabla\phi_\omega \cdot \big((\phi_0+\varepsilon)\nabla f + f\nabla\phi_0\big) + \int_{\partial M} B_1\\
&= \int_M (\phi_0+\varepsilon)\nabla\phi_\omega\cdot\nabla f\\
&\qquad - \int_M \phi_\omega\big(\nabla f\cdot\nabla\phi_0 + f\Delta\phi_0\big) + \int_{\partial M} B_2\\
&= \int_M \big((\phi_0+\varepsilon)\nabla\phi_\omega - \phi_\omega\nabla\phi_0\big)\cdot\nabla f\\
&\qquad + \mu_0\int_M f\phi_0\phi_\omega + \int_{\partial M} B_2\,,
\end{aligned}$$

where $B_i\colon \partial M \to \mathbb{C}$ are boundary functions that will be combined and written explicitly below (equation (4.30)). (We clarify that even though the functions above are $\mathbb{C}$-valued, the notation $\nabla\phi_\omega\cdot\nabla f$ denotes $\sum_i \partial_i\phi_\omega\partial_i f$, and not the complex inner product.)

Similarly, using the fact that ω is harmonic, we have

$$\begin{aligned}
&-4\pi i\int_M (\phi_0+\varepsilon) f\omega\cdot\nabla\phi_\omega\\
&\quad= -2\pi i\int_M (\phi_0+\varepsilon) f\nabla\phi_\omega\cdot\omega\\
&\qquad\quad + 2\pi i\int_M \phi_\omega\big((\phi_0+\varepsilon)\nabla f + f\nabla\phi_0\big)\cdot\omega + \int_{\partial M} B_3\\
&\quad= -2\pi i\int_M \big((\phi_0+\varepsilon)\nabla\phi_\omega - \phi_\omega\nabla\phi_0\big)\cdot(f\omega)\\
&\qquad\quad + 2\pi i\int_M (\phi_0+\varepsilon)\phi_\omega\nabla f\cdot\omega + \int_{\partial M} B_3\,.
\end{aligned}$$

Combining the above, we have

$$\begin{aligned}
\mu_\omega - \mu_0\int_M f\phi_0\phi_\omega &= \int_M \big((\phi_0+\varepsilon)\nabla\phi_\omega - \phi_\omega\nabla\phi_0\big)\cdot\big(\nabla f - 2\pi i f\omega\big)\\
&\quad + \int_M (\phi_0+\varepsilon)\phi_\omega\big(4\pi^2|\omega|^2 f + 2\pi i\nabla f\cdot\omega\big) + \int_{\partial M} B_0\,,
\end{aligned} \tag{4.29}$$

where

$$B_0 = -\overline{\phi_\omega}\partial_\nu\phi_\omega + \phi_\omega f\partial_\nu\phi_0 - 2\pi i(\phi_0+\varepsilon)\phi_\omega f\omega\cdot\nu\,. \tag{4.30}$$

The boundary conditions imposed ensure that $B_0 = 0$ on both $\partial_D M$ and $\partial_N M$.

Since $f = \overline{\phi_\omega}/(\phi_0+\varepsilon)$, we have

$$\nabla f = \frac{(\phi_0+\varepsilon)\nabla\overline{\phi_\omega} - \overline{\phi_\omega}\nabla\phi_0}{(\phi_0+\varepsilon)^2}.$$

Substituting this into the right hand side of (4.29), we obtain a perfect square:

$$\mu_\omega - \mu_0\int_M f\phi_0\phi_\omega = \int_M \Big|2\pi\phi_\omega\omega - \frac{i((\phi_0+\varepsilon)\nabla\phi_\omega - \phi_\omega\nabla\phi_0)}{\phi_0+\varepsilon}\Big|^2\,. \tag{4.31}$$

In particular,

$$\mu_\omega - \mu_0 \int_M f\phi_0\phi_\omega = \mu_\omega - \mu_0 \int_M \frac{\phi_0}{\phi_0+\varepsilon}|\phi_\omega|^2 \geqslant 0.$$

Sending $\varepsilon \to 0$, we obtain $\mu_\omega \geqslant \mu_0$, and so the function $\mathcal{G} \ni \chi \mapsto \lambda_{\chi,0}$ attains global minimum at $\chi = \mathbf{1}$.

To see that $\chi = \mathbf{1}$ is the unique global minimum point, suppose that $\lambda_\chi = \lambda_0$ for some $\chi \in \mathcal{G}$. Writing $\chi = \chi_\omega$ for some $\omega \in \mathcal{H}^1_G$, this means $\mu_\omega = \mu_0$. Fatou's lemma and (4.31) imply

$$\begin{aligned}
&\int_M \Big| 2\pi\phi_\omega\omega - \frac{i\big(\phi_0\nabla\phi_\omega - \phi_\omega\nabla\phi_0\big)}{\phi_0}\Big|^2 \\
&\qquad \leqslant \liminf_{\varepsilon\to 0} \int_M \Big| 2\pi\phi_\omega\omega - \frac{i\big((\phi_0+\varepsilon)\nabla\phi_\omega - \phi_\omega\nabla\phi_0\big)}{\phi_0+\varepsilon}\Big|^2 \\
&\qquad = \mu_\omega - \mu_0 = 0\,,
\end{aligned}$$

by assumption. Hence

$$2\pi\phi_\omega\omega - \frac{i(\phi_0\nabla\phi_\omega - \phi_\omega\nabla\phi_0)}{\phi_0} = 0 \quad \text{in } M\,. \tag{4.32}$$

Since $\phi_\omega = s_{\chi,0}/\sigma_\omega$, we compute

$$\nabla\phi_\omega = \frac{\sigma_\omega\nabla s_{\chi,0} - 2\pi i\sigma_\omega s_{\chi,0}\omega}{\sigma_\omega^2}.$$

Substituting this into (4.32), we see

$$\phi_0\nabla s_{\chi,0} = s_{\chi,0}\nabla\phi_0,$$

which implies that

$$\nabla\Big(\frac{s_{\chi,0}}{\phi_0}\Big) = 0.$$

Therefore, $s_{\chi,0} = c\phi_0$ for some non-zero constant c. However, the twisting conditions (4.1) for ϕ_0 and $s_{\chi,0}$ require

$$\phi_0(g(x)) = \phi_0(x) \qquad \text{and} \qquad s_{\chi,0}(g(x)) = \chi(g)s_{\chi,0}(x)\,,$$

for every $g \in \mathcal{G}$. This is only possible if $\chi(g) = 1$ for all $g \in \mathcal{G}$, showing χ is the trivial representation $\mathbf{1}$. $\square$

4.4. **Positivity of the Hessian (proof of Lemma 4.5).** In this subsection we prove Lemma 4.5. The main difficulty is proving positivity, which we postpone to the end.

Proof of Lemma 4.5. Given $\omega \in \mathcal{H}^1_G$, define

$$\varphi_t = \phi_{t\omega} \qquad \text{and} \qquad h_t = \mu_{t\omega}\,,$$

where $\phi_{t\omega} = \phi_{t\omega,0}$ is the principal eigenfunction of $-H_{t\omega}$ (equation (4.26)) and $\mu_{t\omega}$ is the corresponding principal eigenvalue. We claim that

$$h_0' = 0\,, \quad h_0'' = \mathcal{I}(\omega) \quad \text{and} \quad \operatorname{Re}(\varphi_0') = 0\,, \tag{4.33}$$

where h', φ' denote the derivatives of h and φ respectively with respect to t. This will immediately imply that at $\omega = 0$ the quadratic form induced by the Hessian of the map $\omega \mapsto \mu_\omega$ is precisely $\mathcal{I}(\omega)$, hence proving (4.8) in the lemma.

To establish (4.33), we first note that (4.27) implies

$$-\Delta\varphi_t - 4\pi i t\omega\cdot\nabla\varphi_t + 4\pi^2 t^2|\omega|^2\varphi_t = h_t\varphi_t\,. \tag{4.34}$$

Conjugating both sides of (4.34) gives

$$-\Delta\overline{\varphi_t} - 4\pi i(-t)\omega\cdot\nabla\overline{\varphi_t} + 4\pi^2(-t)^2|\omega|^2\overline{\varphi_t} = h_t\overline{\varphi_t}\,. \tag{4.35}$$

In other words, $\overline{\varphi_t}$ is an eigenfunction of $-H_{-t\omega}$ with eigenvalue h_t. Since $h_t = \mu_{t\omega}$ is the principal eigenvalue, this implies $h_{-t} \leqslant h_t$. By symmetry, we see that $h_{-t} = h_t$, and hence $h_0' = 0$.

To see that φ_0' is purely imaginary, recall h_t is a simple eigenvalue of $-H_{t\omega}$ when t is small. Thus

$$\overline{\varphi_t} = \zeta_t\varphi_{-t}\,, \tag{4.36}$$

for some S^1 valued function ζ_t, defined for small t. Changing t to $-t$, we get

$$\overline{\varphi_{-t}} = \zeta_{-t}\varphi_t = \zeta_{-t}\overline{\zeta_t}\,\overline{\varphi_{-t}}\,.$$

Therefore, $\zeta_{-t}\overline{\zeta_t} = 1$, which implies that $\zeta_{-t} = \zeta_t$. In particular, $\zeta_0' = 0$. Differentiating (4.36) and using the fact that $\zeta_0 = 1$, we get

$$\overline{\varphi_0'} = -\varphi_0'\,,$$

showing that φ_0' is purely imaginary as claimed.

To compute h_0'', we differentiate (4.34) twice with respect to t. At $t = 0$ this gives

$$-\Delta\varphi_0' - 4\pi i\omega\cdot\nabla\varphi_0 = \lambda_0\varphi_0', \tag{4.37}$$

and

$$-\Delta\varphi_0'' - 8\pi i\omega\cdot\nabla\varphi_0' + 8\pi^2|\omega|^2\phi_0 = h_0''\phi_0 + \lambda_0\varphi_0''\,, \tag{4.38}$$

since $\varphi_0 = \phi_0$. Multiplying both sides of (4.38) by ϕ_0 and integrating over M gives

$$h_0'' = \int_M \big(8\pi^2|\omega|^2\phi_0^2 - 8\pi i\phi_0\omega\cdot\nabla\varphi_0'\big)\,. \tag{4.39}$$

Recalling that φ_0' is purely imaginary, we let g_ω be the real valued function defined by $g_\omega = -i\varphi_0'$. Now equation (4.37) shows that g_ω satisfies (2.4). Moreover since $\varphi_0 = 0$ on $\partial_D M$ and $\nu\cdot\nabla\varphi_0 = 0$ on $\partial_N M$, the function g_ω satisfies the boundary conditions (2.5). Therefore, (4.39) reduces to (2.3), showing that $h_0'' = \mathcal{I}(\omega)$ as claimed.

Finally, we show that $\omega\mapsto\mathcal{I}(\omega)$ defined by (2.3) is a well defined positive definite quadratic form on $\mathcal{H}^1_G$. To see that $\mathcal{I}$ is well defined, we first note that in order for (2.4) to have a solution, we need to verify the solvability condition

$$\int_M \phi_0\big(4\pi\omega\cdot\nabla\phi_0\big) = 0\,.$$

This is easily verified as

$$\int_M \phi_0\omega\cdot\nabla\phi_0 = \frac{1}{2}\int_M \omega\cdot\nabla\phi_0^2 = 0\,. \tag{4.40}$$

Hence g_ω is uniquely defined up to the addition of a scalar multiple of ϕ_0 (the kernel of $\Delta+\lambda_0$). Now, using (4.40) again, we see that replacing g_ω with $g_\omega + \alpha\phi_0$ does not change the value of $\mathcal{I}(\omega)$. Thus, $\mathcal{I}(\omega)$ is a well defined function. The fact that $\mathcal{I}$ is a quadratic form (2.3) and the fact that

$$g_{\tau+\omega} = g_\tau + g_\omega \pmod{\phi_0}\,.$$

It remains to show that $\mathcal{I}$ is positive definite. Note that, in view of Lemma 4.4, we already know that $\mathcal{I}$ induces a positive *semi*-definite quadratic form on $\mathcal{H}^1_G$.

For the convenience of notation, let $g = g_\omega = -i\varphi_0'$ as above. As before, we write

$$g = (\phi_0 + \varepsilon) f_\varepsilon \,, \quad \text{where } f_\varepsilon \stackrel{\text{def}}{=} \frac{g}{\phi_0 + \varepsilon} \,.$$

and we will multiply both sides of (2.4) by g and integrate. In preparation for this we compute

$$\begin{aligned} -\int_M (\phi_0 + \varepsilon) f_\varepsilon \Delta g &= \int_M \nabla g \cdot \Bigl(f_\varepsilon \nabla \phi_0 + (\phi_0 + \varepsilon) \nabla f_\varepsilon \Bigr) \\ &= \lambda_0 \int_M \phi_0 f_\varepsilon g - \int_M g \nabla f_\varepsilon \cdot \nabla \phi_0 + \int_M (\phi_0 + \varepsilon) \nabla f_\varepsilon \cdot \nabla g \,, \end{aligned}$$

and

$$\begin{aligned} 4\pi \int_M (\phi_0 + \varepsilon) f_\varepsilon \omega \cdot \nabla (\phi_0 + \varepsilon) &= 2\pi \int_M f_\varepsilon \omega \cdot \nabla (\phi_0 + \varepsilon)^2 \\ &= -2\pi \int_M (\phi_0 + \varepsilon)^2 \nabla f_\varepsilon \cdot \omega \,. \end{aligned}$$

We remark that when integrating by parts above, the boundary terms that arise all vanish because of the boundary conditions imposed. Thus, multiplying (2.4) by $(\phi_0 + \varepsilon) f_\varepsilon$ and integrating gives

$$\begin{aligned} \lambda_0 \int_M g^2 \Bigl(1 - \frac{\phi_0}{\phi_0 + \varepsilon} \Bigr) &= \int_M (\phi_0 + \varepsilon) \nabla f_\varepsilon \cdot \nabla g - \int_M g \nabla f_\varepsilon \cdot \nabla (\phi_0 + \varepsilon) \\ &\quad + 2\pi \int_M (\phi_0 + \varepsilon)^2 \nabla f_\varepsilon \cdot \omega \,. \end{aligned} \tag{4.41}$$

Writing $\tau \stackrel{\text{def}}{=} 2\pi\omega$ and adding the integral

$$J_\varepsilon \stackrel{\text{def}}{=} \int_M (\phi_0 + \varepsilon) \tau \cdot \nabla g - \int_M g \tau \cdot \nabla (\phi_0 + \varepsilon) + \int_M (\phi_0 + \varepsilon)^2 |\tau|^2$$

to both sides of (4.41), we obtain

$$\begin{aligned} J_\varepsilon + \lambda_0 \int_M g^2 \Bigl(1 - \frac{\phi_0}{\phi_0 + \varepsilon} \Bigr) &= \int_M (\phi_0 + \varepsilon) (\nabla f_\varepsilon + \tau) \cdot \nabla g \\ &\quad - \int_M g (\nabla f_\varepsilon + \tau) \cdot \nabla (\phi_0 + \varepsilon) + \int_M (\phi_0 + \varepsilon)^2 (\nabla f_\varepsilon + \tau) \cdot \tau \,. \end{aligned} \tag{4.42}$$

Now, since $g = (\phi_0 + \varepsilon) f_\varepsilon$, we compute

$$\nabla g = f_\varepsilon \nabla (\phi_0 + \varepsilon) + (\phi_0 + \varepsilon) \nabla f_\varepsilon \,.$$

Substituting this into (4.42) gives

$$J_\varepsilon + \lambda_0 \int_M g^2 \Bigl(1 - \frac{\phi_0}{\phi_0 + \varepsilon} \Bigr) = \int_M (\phi_0 + \varepsilon)^2 |\nabla f_\varepsilon + \tau|^2 \geqslant 0 \,. \tag{4.43}$$

Using (2.3) we see

$$\mathcal{I}(\omega) = 8\pi^2 \int_M |\omega|^2 \phi_0^2 + 4\pi \int_M \phi_0 \omega \cdot \nabla g - 4\pi \int_M g \omega \cdot \nabla \phi_0 \,, \tag{4.44}$$

and hence it follows that

$$\lim_{\varepsilon \to 0} J_\varepsilon = \frac{1}{2} \mathcal{I}(\omega) \,.$$

Also by the dominated convergence theorem, the second term on the left hand side of (4.43) goes to zero as $\varepsilon \to 0$. This shows $\mathcal{I}(\omega) \geqslant 0$.

It remains to show $\mathcal{I}(\omega) > 0$ if $\omega \neq 0$. Note that if $\mathcal{I}(\omega) = 0$, then Fatou's lemma and (4.43) imply

$$\int_M \phi_0^2 |\nabla f + \tau|^2 \leqslant \liminf_{\varepsilon \to 0} \Big(J_\varepsilon + \lambda_0 \int_M g^2 \Big(1 - \frac{\phi_0}{\phi_0 + \varepsilon} \Big) \Big) = 0 \,,$$

where $f \stackrel{\text{def}}{=} g/\phi_0$. Therefore $\nabla f + \tau = 0$ in M and hence $\omega = -\nabla f/(2\pi)$. Since $\omega \in \mathcal{H}^1_G \subseteq \mathcal{H}^1$, this forces

$$\Delta f = 0 \quad \text{in } M \,, \qquad \text{and} \qquad \nu \cdot \nabla f = 0 \quad \text{on } \partial M \,.$$

Consequently $\nabla f = 0$, which in turn implies $\omega = 0$. This completes the proof of the positivity of $\mathcal{I}$. ☐

5. **Proof of the winding number asymptotics (Theorem 3.2).**

In this section, we study the long time behaviour of the abelianized winding number of reflected Brownian motion on a manifold M. We begin by using Theorem 2.1 to prove Theorem 3.2 (Section 5.1). Next, in Section 5.2 we discuss the connection of our results with the work by Toby and Werner [TW95]. Finally, in Section 5.3, modulo certain amount of technicalities which need to be verified, we propose a (sketched) independent probabilistic proof of Theorem 3.2.

5.1. **Proof of Theorem 3.2.** We obtain the long time behaviour of the abelianized winding of reflected Brownian motion in M by applying Theorem 2.1 in this context. Let $\hat{M}$ be a covering space of M with deck transformation group[3] $\pi_1(M)_{\text{ab}}$. In view of the covering factorization (4.18), we may, without loss of generality, assume that $\operatorname{tor}(\pi_1(M)_{\text{ab}}) = \{0\}$. Note that since the deck transformation group $G = \pi_1(M)_{\text{ab}}$ by construction, we have $\mathcal{H}^1_G = \mathcal{H}^1$. Given $n \in \mathbb{Z}^k$ ($k = \operatorname{rank}(\mathrm{G})$), define $g_n \in G$ by

$$g_n \stackrel{\text{def}}{=} \sum_{i=1}^k n_i \pi_G(\gamma_i) \,, \quad \text{where } n = (n_1, \dots, n_k) \in \mathbb{Z}^k \,,$$

where $\{\pi_G(\gamma_1), \dots, \pi_G(\gamma_k)\}$ is the basis of G chosen in Section 3. Clearly $n \mapsto g_n$ is an isomorphism between G and $\mathbb{Z}^k$.

Lemma 5.1. *For any $x, y \in \hat{M}$ and $n \in \mathbb{Z}^k$ we have*

$$d_{\mathcal{I}}(x, g_n(y))^2 = (A^{-1} n) \cdot n + O(|n|) \,,$$

where A is the matrix $(a_{i,j})$ defined by

$$a_{i,j} \stackrel{\text{def}}{=} \langle \omega_i, \omega_j \rangle_{\mathcal{I}} = \frac{8\pi^2}{\operatorname{vol}(M)} \int_M \omega_i \cdot \omega_j \,. \tag{5.1}$$

Proof. Given $\omega \in \mathcal{H}^1$ we compute

$$\xi_{x, g_n(y)}(\omega) = \int_x^y \boldsymbol{\pi}^*(\omega) + \int_y^{g_n(y)} \boldsymbol{\pi}^*(\omega) \,, \tag{5.2}$$

[3] The existence of such a cover is easily established by taking the quotient of the universal cover $\bar{M}$ by the action of the commutator of $\pi_1(M)$.

where the integrals are taken along any smooth path in $\hat{M}$ connecting the endpoints. Note that the integrals are well defined, and the second one is independent of y. Therre, if for any $g \in G$ we define $\psi_g \colon \mathcal{H}^1 \to \mathbb{R}$ by

$$\psi_g(\omega) = \int_y^{g(y)} \boldsymbol{\pi}^*(\omega)\,,$$

then (5.2) becomes

$$\xi_{x,g_n(y)}(\omega) = \xi_{x,y}(\omega) + \psi_{g_n}(\omega)\,.$$

It follows that

$$d_{\mathcal{I}}(x, g_n(y))^2 = d_{\mathcal{I}}(x,y)^2 + \sum_{i=1}^k n_i \langle \psi_{\pi_G(\gamma_i)}, \xi_{x,y} \rangle_{\mathcal{I}^*} + \sum_{i,j=1}^k n_i n_i \langle \pi_G(\gamma_i), \pi_G(\gamma_j) \rangle_{\mathcal{I}^*}\,.$$

Since $\{\omega_1, \dots, \omega_k\}$ is the dual basis to $\{\pi_G(\gamma_1), \dots, \pi_G(\gamma_j)\}$, we have

$$\langle \pi_G(\gamma_i), \pi_G(\gamma_j) \rangle_{\mathcal{I}^*} = (A^{-1})_{i,j}\,.$$

Therefore, the result follows. Note that the second equality of (5.1) follows from the fact that (2.10) holds under Neumann boundary conditions. □

Now we prove Theorem 3.2.

Proof of Theorem 3.2. Recall in Section 3 we decomposed the universal cover $\bar{M}$ as the disjoint union of fundamental domains $\bar{U}_g$ indexed by $g \in \pi_1(M)$. Projecting these domains to the cover $\hat{M}$ we write $\hat{M}$ as the disjoint union of fundamental domains $\bar{U}_g$ indexed by $g \in G$. Let $\hat{W}$ be the lift of the trajectory of W to $\hat{M}$, and observe that if $\hat{W}(t) \in \hat{U}_{g_n}$, then $\rho(t) = n$.

We use this to compute the characteristic function of $\rho(t)/\sqrt{t}$ as follows. Since the generator of $\hat{W}$ is $\frac{1}{2}\Delta$, its transition density is given by $\hat{H}(t/2, \cdot, \cdot)$. Hence, for any $z \in \mathbb{R}^k$ we have

$$\begin{aligned}\boldsymbol{E}^x \Big[\exp\Big(\frac{iz \cdot \rho(t)}{t^{1/2}}\Big)\Big] &= \sum_{n \in \mathbb{Z}^k} \exp\Big(\frac{iz \cdot n}{t^{1/2}}\Big) \boldsymbol{P}^x(\hat{W}(t) \in \hat{U}_{g_n}) \\ &= \sum_{n \in \mathbb{Z}^k} \int_{\hat{U}_{g_n}} \hat{H}\Big(\frac{t}{2}, x, y\Big) \exp\Big(\frac{iz \cdot n}{t^{1/2}}\Big)\, dy\,.\end{aligned}$$

By Theorem 2.1 and Remark 2.3, this means that uniformly in $x \in \hat{M}$ we have

$$\begin{aligned}&\lim_{t \to \infty} \boldsymbol{E}^x \Big[\exp\Big(\frac{iz \cdot \rho(t)}{t^{1/2}}\Big)\Big] \\ &\quad = C_{\mathcal{I}} \lim_{t \to \infty} \sum_{n \in \mathbb{Z}^k} \int_{\hat{U}_{g_n}} \frac{2^{k/2}}{t^{k/2}} \exp\Big(-\frac{4\pi^2 d_{\mathcal{I}}(x, g_n(y))^2}{t} + \frac{iz \cdot n}{t^{1/2}}\Big)\, dy \\ &\quad = C_{\mathcal{I}} \lim_{t \to \infty} \sum_{n \in \mathbb{Z}^k} \frac{2^{k/2}}{t^{k/2}} \exp\Big(-\frac{4\pi^2 (A^{-1}n) \cdot n}{t} + \frac{iz \cdot n}{t^{1/2}}\Big)\,,\end{aligned}$$

where the last equality followed from Lemma 5.1 above. Observe that the last term is the Riemann sum of a standard Gaussian integral. Therefore,

$$\lim_{t \to \infty} \boldsymbol{E}^x \Big[\exp\Big(\frac{iz \cdot \rho(t)}{t^{1/2}}\Big)\Big] = 2^{k/2} C_{\mathcal{I}} \int_{\zeta \in \mathbb{R}^k} \exp\Big(-4\pi^2 (A^{-1}\zeta) \cdot \zeta + iz \cdot \zeta\Big)\, d\zeta\,.$$

This shows that as $t \to \infty$, $\rho(t)/\sqrt{t}$ converges to a normally distributed random variable with mean 0 and covariance matrix $A/(8\pi^2)$. By (3.3) and (5.1) we see that $\Sigma = A/(8\pi^2)$, which completes the proof of the second assertion in (3.1) of the theorem. The first assertion follows immediately from the second assertion and Chebychev's inequality. ☐

5.2. **Relation to the work of Toby and Werner.** Toby and Werner [TW95] studied the long time behaviour of the winding of an obliquely reflected Brownian motion in bounded planar domains. In this case, we describe their result and relate it to Theorem 3.2.

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with k holes $V_1, \cdots, V_k$ of positive volume. Let W_t be a reflected Brownian motion in Ω with a non-tangential reflecting vector field $u \in C^1(\partial\Omega)$. Let $p_1, \cdots, p_k$ be k distinct points in $\mathbb{R}^2$. For $1 \leqslant j \leqslant k$, define $\rho(t, p_j)$ to be the winding number of W_t with respect to the point p_j.

Theorem 5.2 (Toby and Werner, 1995)**.** *There exist constants a_i, b_i, depending on the domain Ω, such that*

$$\frac{1}{t}\big(\rho(t,p_1), \cdots, \rho(t,p_k)\big) \xrightarrow[t\to\infty]{w} \big(a_1 C_1 + b_1, \cdots, a_k C_k + b_k\big)\,, \tag{5.3}$$

where C_1, ..., C_k are standard Cauchy variables. Moreover, for any j such that $p_j \notin \Omega$, a_j must be equal to zero.

When $p_j \in \Omega$, the process W can wind a large number of times around p_j in a short period as it approaches p_j. This is why the heavy-tailed Cauchy distribution arises in Theorem 5.2, and the limiting process is non-degenerate precisely when each $p_j \in \Omega$. This is exactly the situation when Theorem 5.2 is sharp.

On the other hand, if $p_j \in V_j$, we have $a_j = 0$ and (5.3) becomes a law of large numbers. In the case with normal reflection, Theorem 3.2 provides the central limit theorem for the fluctuation around the mean. Therefore, in this case our result is a refinement of Theorem 5.2.

It is not pointed out nor can be easily seen from [TW95] why the mean $b_j = 0$ in the normal reflection case with $p_j \in V_j$. For completeness, we give a proof of this fact below.

Recall that (see for instance Stroock-Varadhan [SV71]) reflected Brownian motion has the semi-martingale representation

$$W_t = \beta_t + \int_0^t u(W_s)\, dL_s\,, \tag{5.4}$$

where β_t is a two dimensional Brownian motion, u is the reflecting vector field on $\partial\Omega$, and L_t is a continuous increasing process which increases only when $W_t \in \partial\Omega$. We also know that the process W_t has a unique invariant measure, which is denoted by μ. From [TW95], the constants b_j are given by

$$b_j = \frac{1}{2\pi} \int_{p\in\Omega} \boldsymbol{E}^p \Big[\int_0^1 u_j(W_s) dL_s\Big]\, d\mu(p)\,, \tag{5.5}$$

where $u_j : \partial\Omega \to \mathbb{R}$ is defined by

$$u_j(p) \stackrel{\text{def}}{=} \frac{u(p)\cdot(p-p_j)^\perp}{|p-p_j|}\,,$$

and $q^\perp \stackrel{\text{def}}{=} (-q_2, q_1)$ for $q = (q_1, q_2) \in \mathbb{R}^2$.

Proposition 5.3. *Let W_t be a normally reflected Brownian motion in Ω, and $p_j \in V_j$ for each j. Then $b_j = 0$ for all j, and consequently*

$$\lim_{t\to\infty} \frac{\rho(t,p_j)}{t} \xrightarrow{P} 0\,.$$

Proof. Fix $1 \leqslant j \leqslant k$. Let $w(t,p)$ be the solution to the following initial-boundary value problem:

$$\begin{cases} \partial_t w - \dfrac{1}{2}\Delta w = 0 & \text{in } (0,\infty)\times\Omega\,,\\ \nu\cdot\nabla w = -u_j & \text{on } (0,\infty)\times\partial\Omega\,,\\ \lim\limits_{t\to 0} w(t,\cdot) = 0 & \text{in } \Omega\,, \end{cases} \tag{5.6}$$

where ν is the outward pointing unit normal on the boundary. By applying Itô's formula to the process $[0,t-\varepsilon] \ni s \mapsto w(t-s, W_s)$ and using the semi-martingale representation (5.4) of W_t, we get

$$\begin{aligned} w(t,p) - \boldsymbol{E}^p\Big[w(\varepsilon, W_{t-\varepsilon})\Big] &= -\boldsymbol{E}^p\Big[\int_0^{t-\varepsilon} \nu\cdot\nabla w(W_s, t-s)dL_s\Big]\\ &= \boldsymbol{E}^p\Big[\int_0^{t-\varepsilon} u_j(W_s)dL_s\Big], \end{aligned}$$

where in the last identity we have used the fact that dL_s is carried by the set $\{s \geqslant 0 : W_s \in \partial\Omega\}$. Since $\boldsymbol{P}(B_t \in \partial U) = 0$, sending $\varepsilon \to 0$ and using the dominated convergence theorem gives

$$w(t,p) = \boldsymbol{E}^p\Big[\int_0^t u_j(W_s)dL_s\Big]\,.$$

On the other hand, according to Harrison, Landau and Shepp [HLS85], Theorem 2.8, the invariant measure μ of W_t is the unique probability measure on the closure $\bar\Omega$ of Ω that $\mu(\partial\Omega) = 0$ and

$$\int_\Omega \Delta f(p)\,d\mu(p) \leqslant 0 \quad \text{for all } f \in C^2(\bar\Omega) \text{ with } \nu\cdot\nabla f \leqslant 0 \text{ on } \partial\Omega.$$

Stokes' theorem now implies μ is the normalized Lebesgue measure on Ω. Consequently,

$$b_j = \frac{1}{2\pi\,\mathrm{vol}(\Omega)} \int_\Omega \boldsymbol{E}^p\Big[\int_0^1 u_j(W_s)\,dL_s\Big]\,dp = \frac{1}{2\pi\,\mathrm{vol}(\Omega)}\int_\Omega w(1,p)\,dp\,.$$

Integrating (5.6) over Ω and using the boundary conditions yields

$$\begin{aligned} 0 &= \partial_t \int_\Omega w\,dp - \int_\Omega \Delta w\,dp\\ &= \partial_t \int_\Omega w\,dp + \int_{\partial\Omega} u_j(p)\,dp\\ &= \partial_t \int_\Omega w\,dp - \int_{\partial\Omega} \nu\cdot\frac{(p-p_j)^\perp}{|p-p_j|}\,dp\,. \end{aligned}$$

Since when $p_j \in V_j$ the vector field $p \mapsto (p-p_j)^\perp/|p-p_j|$ is a divergence free vector field on $\bar\Omega$, the last integral above above vanishes. Thus

$$\partial_t \int_\Omega w\,dp = 0\,,$$

and since $w = 0$ when $t = 0$, $w = 0$ for all $t \geqslant 0$, and hence $b_j = 0$. $\square$

5.3. **A Probabilistic Proof of Theorem 3.2.** As mentioned earlier, Theorem 3.2 can also be proved by using a probabilistic argument. Modulo certain technicalities, we sketch this argument below.

First suppose $\gamma\colon [0,\infty) \to M$ is a smooth path. Let $\rho(t,\gamma)$ be the $\mathbb{Z}^k$-valued winding number of γ, as in Definition 3.1. Namely, let $\bar{\gamma}$ be the lift of γ to the universal cover of M, and let $\rho(t,\gamma) = (n_1, \dots, n_k)$ if

$$\pi_G\big(\bar{\boldsymbol{g}}(\bar{\gamma}(t))\big) = \sum_{i=1}^{k} n_i \pi_G(\gamma_i)\,.$$

By our choice of $(\omega_1, \dots, \omega_k)$ we see that $\rho_i(t,\gamma)$, the i^{th} component of $\rho(t,\gamma)$, is precisely the integer part of $\theta_i(t,\gamma)$, where

$$\theta_i(t,\gamma) \stackrel{\text{def}}{=} \int_{\gamma([0,t])} \omega_i = \int_0^t \omega_i(\gamma(s))\,\gamma'(s)\,ds\,. \tag{5.7}$$

If M is a planar domain with k holes, and the forms ω_i are chosen as in Remark 2.4, then $2\pi\theta_i(t,\gamma)$ is the total angle γ winds around the k^{th} hole up to time t.

In the case when γ is not smooth, the theory of rough paths can be used to give meaning to the above path integrals. In particular, when γ is the trajectory of semimartingale on M, we know that the integral obtained via the theory of rough paths agrees with the Stratonovich integral. To fix notation, let W be a reflected Brownian motion in M, and $\rho(t) = (\rho_1(t), \cdots, \rho_k(t))$ to be the $\mathbb{Z}^k$-valued winding number of W as in Definition 3.1. Then we must have $\rho_i(t) = \lfloor \theta_i(t) \rfloor$, where $\theta_i(t)$ is the rough path integral, or equivalently, the Stratonovich integral

$$\theta_i(t) = \int_0^t \omega_i(W_s) \circ dW_s\,. \tag{5.8}$$

In Euclidean domains, the long time behaviour of this integral can be obtained as follows. The key point is that the forms ω_i are chosen to be harmonic in M and tangential on ∂M. Consequently, using the semi-martingale decomposition (5.4), we see that

$$\begin{aligned}\theta_i(t) &= \int_0^t \omega_i(W_s) \circ d\beta_s + \int_0^t (\omega_i(W_s) \cdot u(W_s))\,dL_s\\ &= \int_0^t \omega_i(W_s) d\beta_s.\end{aligned}$$

In particular, θ is indeed a martingale with quadratic variation given by

$$\langle \theta_i, \theta_j \rangle_t = \int_0^t \omega_i(W_s) \cdot \omega_j(W_s)\,ds\,. \tag{5.9}$$

Moreover, by Harrison et. al. [HLS85], the unique invariant measure of W_t is the normalized Lebesgue measure. Therefore, according to the ergodic theorem,

$$\lim_{t\to\infty} \frac{1}{t} \langle \theta_i, \theta_j \rangle_t = \frac{1}{\operatorname{vol}(M)} \int_M \omega_i \cdot \omega_j$$

for almost surely. Now we can conclude from the martingale central limit theorem (see [PS08, Theorem 3.33 and Corollary 3.34]) that

$$\frac{\theta_t}{\sqrt{t}} \xrightarrow[w]{t\to\infty} \mathcal{N}(0, \Sigma)\,,$$

where the covariance matrix Σ is given by (3.3).

To extend the above argument to the geometric setting, one first needs to establish the analogue of the semi-martingale decomposition (5.4) on manifolds with boundary. While this should be a technical adaptation of [SV71], there is no easily available reference. In addition, one needs to to show that θ_i is a martingale with quadratic variation (5.9). This might be done through a localization argument by breaking the Stratonovich integral defining θ_i (equation (5.8)) into pieces that are entirely contained in local coordinate charts, and using the analogue of (5.4) together with the fact that $\omega \in \mathcal{H}^1$. Now the other parts of the argument should be the same as the Euclidean case.

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